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A Corner Point Gibbs Phenomenon for Fourier Series in Two Dimensions

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Let f be the function periodic with period 2π in x and y which extends the indicator function of the parallelogram $A = \{(x, y) : 0 \leqslant y \leqslant \pi,\ y/c \leqslant x \leqslant y/c + \pi\}$ $(0 \neq c \in \mathbb{R})$. The partial sums of the Fourier series of f of order $2M+1$, say, evaluated at $(\pi x/(2M+1), \pi y/(2M+1))$, converge for $M \to \infty$ to a sum of integrals of the functions $\sin t/t$, $\sin s/s \sin t/t$, $\cos s/s \cos t/t$ over domains depending on x y, and c. This limit appears to depend only on the part of A inside an arbitrarily small circle about 0.

1. THE MAIN RESULT.

Fourier series of functions in two variables enjoy convergence properties similar to those of Fourier series in one variable [16, Chap. XVII, 1]. In order to investigate the appearance of a Gibbs phenomenon in the two-dimensional situation it seems suitable to consider functions with simple and typical discontinuities. "Edge point" discontinuities have already been studied by various authors; a "corner point" discontinuity of a function on the sphere, expanded into a series of spherical harmonics, has been studied by Weyl [14, 15] (cf. Remark 3). Characteristic for the Gibbs phenomenon is the persistence of over- and undershoots of the partial sums of the Fourier series close to the jump discontinuity. More insight in the phenomenon is obtained if the partial sums are evaluated in neighbourhoods of the discontinuity, rescaled proportionally to the order of the partial sums.

The purpose of the present note is to study the Gibbs phenomenon, located at the "corner point" (0, 0), for the Fourier series of the function f with period 2π in x and y which extends the indicator function 1_A of the set

$$A = \{(x, y) : 0 \leqslant y \leqslant \pi,\ y/c \leqslant x \leqslant y/c + \pi\} \qquad (0 \neq c \in \mathbb{R}).$$

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Standard calculations furnish the Fourier series

$$f\sim\frac{1}{4}+\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)\,y}{2m+1}+\frac{1}{\pi}\sum_{\substack{m=0\\(2m+1)/c\in\mathbb{Z}}}^{\infty}\frac{\sin(2m+1)(x-y/c)}{2m+1}$$

$$+\frac{2}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sum_{\substack{l=-\infty\\ l\neq-(2m+1)/c}}^{\infty}\frac{\sin\frac{\pi}{2}\left(\frac{(2m+1)}{c}+l\right)}{\frac{(2m+1)}{c}+l}$$

$$\times\sin\left((2m+1)\,x+ly-\frac{\pi}{2}\left(\frac{(2m+1)}{c}+l\right)\right)$$

$$=\frac{1}{4}+\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)\,y}{2m+1}+\frac{1}{\pi}\sum_{\substack{m=0\\(2m+1)/c\in\mathbb{Z}}}^{\infty}\frac{\sin(2m+1)(x-y/c)}{2m+1}$$

$$+\frac{1}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{2m+1}\sum_{\substack{l=-\infty\\ l\neq-(2m+1)/c}}^{\infty}\left\{\frac{\sin\pi\left(\frac{(2m+1)}{c}+l\right)}{\frac{(2m+1)}{c}+l}\sin((2m+1)\,x+ly)\right.$$

$$\left.-2\,\frac{\sin^2\frac{\pi}{2}\left(\frac{(2m+1)}{c}+l\right)}{\frac{(2m+1)}{c}+l}\cos((2m+1)\,x+ly)\right\}.$$

Note that the terms of the second sum appear formally as the limits of the corresponding terms of the last sum as $l\to-(2m+1)/c$.

The mentioned Gibbs phenomenon becomes apparent if the behaviour of partial sums of order $2M+1$, say, in m and l is observed in the point

$$P=\left(x=\frac{\pi\left(x_1+\frac{y_1}{c}\right)}{2M+1},\ y=\frac{\pi y_1}{2M+1}\right); \tag{1}$$

here x_1 and y_1 respectively determine the distances of P from the extended sides of A, measured in units of size $\pi/(2M+1)$. We interpret "of order $2M+1$" as

$$\sum_{m=-M}^{M}\ \sum_{l=-(2M+1)}^{2M+1}c_{2m+1,\,l}\,e^{((2m+1)\,x+ly)\,i}. \tag{2}$$

Note that the conditions $0 \leqslant m \leqslant M$ and $|l| \leqslant 2M+1$ impose on the terms of the partial sum (2) satisfying $l = -(2m+1)/c$ the restriction

$$m \leqslant M_c = |c|\, M + \frac{|c|-1}{2}. \tag{3}$$

Correspondingly we shall use the notation

$$\bar{M} = \min(M, M_c) = \begin{cases} M & \text{for} \quad |c| > 1 \\ M_c & \text{for} \quad |c| \leqslant 1 \end{cases} \tag{4}$$

and we shall write $\sum_{m=0}^{\bar{M}}$ for $\sum_{m=0}^{[\bar{M}]}$.

According to (1), let

$$s_M(x_1, y_1, c) = \frac{1}{4} + \sum_{j=1}^{6} s_M^{(j)}(x_1, y_1, c),$$

where

$$s_M^{(1)}(x_1, y_1, c) = s_M^{(1)}(y_1) = \frac{1}{\pi} \sum_{m=0}^{M} \frac{\sin \pi \dfrac{2m+1}{2M+1}\, y_1}{2m+1}$$

$$s_M^{(2)}(x_1, y_1, c) = s_M^{(2)}(x_1, c) = \frac{1}{\pi} \sum_{\substack{m=0 \\ (2m+1)/c \in \mathbb{Z}}}^{\bar{M}} \frac{\sin \pi \dfrac{2m+1}{2M+1}\, x_1}{2m+1}$$

$$s_M^{(3)}(x_1, y_1, c) = \frac{1}{\pi^2} \sum_{\substack{m=0 \\ (2m+1)/c \notin \mathbb{Z}}}^{M} \frac{\sin \pi \dfrac{2m+1}{2M+1}\, x_1}{2m+1} \times \sum_{l=(2M+1)}^{2M+1} \sin \pi \left(\frac{2m+1}{c} + l\right) \frac{\cos \pi \dfrac{(2m+1)/c + l}{2M+1}\, y_1}{\dfrac{2m+1}{c} + l}$$

$$s_M^{(4)}(x_1, y_1, c) = \frac{1}{\pi^2} \sum_{\substack{m=0 \\ (2m+1)/c \notin \mathbb{Z}}}^{M} \frac{\cos \pi \dfrac{2m+1}{2M+1}\, x_1}{2m+1} \times \sum_{l=-(2M+1)}^{2M+1} \sin \pi \left(\frac{2m+1}{c} + l\right) \frac{\sin \pi \dfrac{(2m+1)/c + l}{2M+1}\, y_1}{\dfrac{2m+1}{c} + l}$$

$$s_M^{(5)}(x_1, y_1, c) = \frac{2}{\pi^2} \sum_{m=0}^{M} \frac{\sin \pi \dfrac{2m+1}{2M+1} x_1}{2m+1}$$

$$\times \sum_{\substack{l=-(2M+1) \\ l \neq -(2m+1)/c}}^{2M+1} \sin^2 \frac{\pi}{2}\left(\frac{2m+1}{c} + l\right) \frac{\sin \pi \dfrac{(2m+1)/c + l}{2M+1} y_1}{\dfrac{2m+1}{c} + l}$$

$$s_M^{(6)}(x_1, y_1, c) = -\frac{2}{\pi^2} \sum_{m=0}^{M} \frac{\cos \pi \dfrac{2m+1}{2M+1} x_1}{2m+1}$$

$$\times \sum_{\substack{l=-(2M+1) \\ l \neq -(2m+1)/c}}^{2M+1} \sin^2 \frac{\pi}{2}\left(\frac{2m+1}{c} + l\right) \frac{\cos \pi \dfrac{(2m+1)/c + l}{2M+1} y_1}{\dfrac{2m+1}{c} + l}.$$

We shall see in Theorem 1 that the limit

$$s(x_1, y_1, c) = \lim_{M \to \infty} s_M(x_1, y_1, c) \tag{5}$$

exists for every $(x_1, y_1) \in \mathbb{R}^2$ and that, for $x_1 > 0$, $y_1 > 0$ and

$$\bar{c} = \min(|c|, 1),$$

the surface $z = s(x_1, y_1, c)$ governing the Gibbs phenomenon is given by

$$s(x_1, y_1, c) = \frac{1}{4} + \frac{1}{2\pi} \int_0^{\pi y_1} \frac{\sin s}{s}\, ds + \frac{1}{2\pi} \int_0^{\pi x_1 \bar{c}} \frac{\sin t}{t}\, dt$$

$$+ \frac{1}{2\pi^2} \int_0^{\pi x_1} \frac{\sin t}{t}\, dt \int_{(ty_1/cx_1) - \pi y_1}^{(ty_1/cx_1) + \pi y_1} \frac{\sin s}{s}\, ds$$

$$- \frac{1}{2\pi^2} \left\{ \int_0^{\pi x_1 \bar{c}} \frac{\cos t}{t}\, dt \int_{\pi y_1 - (ty_1/cx_1)}^{\pi y_1 + (ty_1/cx_1)} \frac{\cos s}{s}\, ds \right.$$

$$\left. + \int_{\pi x_1 \bar{c}}^{\pi x_1} \frac{\cos t}{t}\, dt \int_{(ty_1/cx_1) - \pi y_1}^{(ty_1/cx_1) + \pi y_1} \frac{\cos s}{s}\, ds \right\}.$$

The set A is chosen in form of a parallelogram only for simplifying the computation of the Fourier series. In truth the just mentioned Gibbs phenomenon is locally determined: replacing A by its intersection with an arbitrarily small circular disk about $(0, 0)$ does not change the above mentioned limit. This fact requires an explicit proof, given in Section 3, since in two dimensions we cannot rely on a general localization principle (cf. [16, Chap. XVII, 1.25]).

THEOREM 1. *Let* $c \in \mathbb{R}$, $c \neq 0$, *and* $(x_1, y_1) \in \mathbb{R}^2$ *be given. Then for* $\bar{c} = \min(|c|, 1)$ *one has*

(a) $$\lim_{M \to \infty} s_M^{(1)}(y_1) = \frac{1}{2\pi} \int_0^{\pi y_1} \frac{\sin s}{s}\, ds;$$

(b) $$\lim_{M \to \infty} (s_M^{(2)}(x_1, c) + s_M^{(3)}(x_1, y_1, c)) = \frac{1}{2\pi} \int_0^{\pi x_1 \bar{c}} \frac{\sin t}{t}\, dt;$$

(c) $$\lim_{M \to \infty} s_m^{(4)}(x_1, y_1, c) = 0;$$

(d) $$\lim_{M \to \infty} s_M^{(5)}(x_1, y_1, c) = \begin{cases} \dfrac{1}{2\pi^2} \displaystyle\int_0^{\pi x_1} \frac{\sin t}{t}\, dt \int_{(ty_1/cx_1) - \pi y_1}^{(ty_1/cx_1) + \pi y_1} \frac{\sin s}{s}\, ds & \textit{for} \quad x_1 \neq 0, \\ 0 & \textit{for} \quad x_1 = 0; \end{cases}$$

(e_1) $$\lim_{M \to \infty} s_M^{(6)}(x_1, y_1, c) = -\frac{1}{2\pi^2} \left\{ \int_0^{\pi x_1 \bar{c}} \frac{\cos t}{t}\, dt \int_{\pi y_1 - (ty_1/cx_1)}^{\pi y_1 + (ty_1/cx_1)} \frac{\cos s}{s}\, ds + \int_{\pi x_1 \bar{c}}^{\pi x_1} \frac{\cos t}{t}\, dt \int_{(ty_1/cx_1) - \pi y_1}^{(ty_1/cx_1) + \pi y_1} \frac{\cos s}{s}\, ds \right\}$$

for $x_1 \neq 0$ *and* $y_1 \neq 0$;

(e_2) $$\lim_{M \to \infty} s_M^{(6)}(x_1, 0, c) = -\frac{1}{2\pi^2} \left\{ \int_0^{\pi x_1 \bar{c}} \frac{\cos t}{t} \log\left(\frac{c\pi x_1 + t}{c\pi x_1 - t}\right) dt + \int_{\pi x_1 \bar{c}}^{\pi x_1} \frac{\cos t}{t} \log\left(\frac{t + c\pi x_1}{t - c\pi x_1}\right) dt \right\}$$

for $x_1 \neq 0$;

(e$_3$) $\lim_{M\to\infty} s_M^{(6)}(0, y_1, c)$

$$= -\frac{1}{2\pi^2}\left\{\int_0^{\bar{c}} \frac{dt}{t} \int_{\pi y_1-(t\pi y_1/c)}^{\pi y_1+(t\pi y_1/c)} \frac{\cos s}{s}\, ds + \int_{\bar{c}}^1 \frac{dt}{t} \int_{(t\pi y_1/c)-\pi y_1}^{(t\pi y_1/c)+\pi y_1} \frac{\cos s}{s}\, ds\right\};$$

for $y_1 \neq 0$

(e$_4$) $$\lim_{M\to\infty} s_M^{(6)}(0, 0, c) = -\frac{1}{2\pi^2}\left\{\int_0^{\bar{c}} \frac{1}{t}\log\left(\frac{c+t}{c-t}\right) dt + \int_{\bar{c}}^1 \frac{1}{t}\log\left(\frac{t+c}{t-c}\right) dt\right\}.$$

Finally we shall investigate the behaviour of the corner point Gibbs phenomenon for $c \searrow 0$ and for $c \nearrow 0$. As is to be expected and as will be stated in Theorem 3 the Gibbs phenomenon will—in the sequence of rescaled neighbourhoods of 0—vanish in the first case and reduce to the one-dimensional Gibbs phenomenon in the second case.

2. PROOF OF THEOREM 1

Because of the symmetry properties of the expressions appearing in Theorem 1 it will suffice to prove the theorem for non-negative values of x_1, y_1, and c. The validity of assertion (b) in case $y_1 = 0$ will be shown separately in the proof of assertion (b) and the equation $\lim_{M\to\infty} s_M^{(4)}(0, y_1, c) = 0$ will be dealt with in the proof of assertion (c).

At some places there will be a separate discussion of cases in which the parameter c assumes special rational values. Avoiding this by a continuity argument would require the justification of the interchange of limits, e.g., for $c \to c_0$ and $M \to \infty$. Still, a second look at the especially simple cases $c = 1/N$ and $c = 2/N$ ($N \in \mathbb{N}$) might convey to the interested reader a feeling for the ideas at the basis of this investigation.

At the outset we make sure of the convergence of the two double integrals in assertion (e), where the integrand has poles in $(t, s) = (0, \pi y_1)$ and, for $c \leqslant 1$, in $(t, s) = (\pi x_1 c, 0)$. This is a consequence of the following Lemma 1, the proof of which is left to the reader.

LEMMA 1. *Let* $a \neq 0$ *Then*

$$\int_0^{\eta} \frac{dt}{t} \int_{a-bt}^{a+bt} \frac{ds}{s} = O(\eta) \qquad as \quad \eta \searrow 0.$$

Proof of Assertion (a). The assertion may be shown by an argument [12] which also illuminates the background of the later considerations. We have

$$s_M^{(1)}(y_1)=\frac{1}{2\pi}\sum_{m=0}^{M}\frac{\sin\pi\dfrac{2m+1}{2M+1}y_1}{\pi\dfrac{2m+1}{2M+1}y_1}\,\frac{2\pi y_1}{2M+1}.$$

As a Riemann sum of a continuous function this converges for $M\to\infty$ to the indicated integral. ∎

We shall several times have opportunity to use a slight quantitative refinement of this argument, the proof of which is again left to the reader.

Riemann Sum Lemma. *Let the real function g be continuous on* $[a, b]$ *and let w be the continuity modulus of g on* $[a, b]$. *Let* $p\in\mathbb{R}$ *and* $0<q\in\mathbb{R}$. *Then for* $[\tau_1, \tau_2]\subset[a, b]$ *and for*

$$t_1=\min\{p+kq\colon p+kq\in[\tau_1, \tau_2]\},$$
$$t_2=\max\{p+kq\colon p+kq\in[\tau_1, \tau_2]\}$$

one has

$$\left|\sum_{\tau_1\leqslant p+kq\leqslant\tau_2} g(p+kq)\cdot q-\int_{\tau_1}^{\tau_2} g(t)\,dt\right|$$
$$\leqslant w(q)(\tau_2-\tau_1)+q\cdot\min(|g(t_1)|, |g(t_2)|).$$

Before entering in the proof of assertion (b), under the presupposition $(2m+1)/c\notin\mathbb{Z}$ we shall study sums of the form

$$S_m(L_1, L_2)=\frac{1}{\pi}\sum_{l=L_1}^{L_2}\sin\pi\left(\frac{2m+1}{c}+l\right)\frac{\cos\pi\dfrac{(2m+1)/c+l}{2M+1}y_1}{\dfrac{2m+1}{c}+l},$$

where $-(2M+1)\leqslant L_1\leqslant L_2\leqslant 2M+1$. In order to facilitate the notation let

$$\lambda_m=\frac{2m+1}{c},$$
$$\lambda_m^0=\lambda_m-[\lambda_m].$$

By our presupposition we have $\lambda_m^0 \neq 0$. Writing $h = l + [\lambda_m]$ we obtain

$$S_m(L_1, L_2) = \frac{1}{\pi} \sin(\pi\lambda_m^0) \sum_{h=H_1}^{H_2} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h}, \tag{6}$$

where

$$-(2M+1) + [\lambda_m] \leqslant H_1 = L_1 + [\lambda_m] \leqslant h \leqslant H_2 = L_2 + [\lambda_m]$$
$$\leqslant 2M + 1 + [\lambda_m].$$

The denominators $\lambda_m^0 + h$ are positive for $h \geqslant 0$ and negative for $h < 0$; these last denominators can turn up only for $-(2M+1) + [\lambda_m] < 0$, i.e., for

$$m < cM + \frac{c-1}{2} = M_c \qquad \text{(cf. (3))}.$$

This condition is satisfied for all $m \leqslant M$ if $c > 1$ but imposes restrictions on m if $c \leqslant 1$.

We shall pay special attention to the sum $S_m(-(2M+1), 2M+1)$ and to decompositions thereof into sums of the types

$$S_m^- = \frac{1}{\pi} \sin(\pi\lambda_m^0) \sum_{h=-(2M+1)+[\lambda_m]}^{-[\eta(2M+1)]-1} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h},$$

$$S_m^+ = \frac{1}{\pi} \sin(\pi\lambda_m^0) \sum_{h=[\eta(2M+1)]}^{2M+1+[\lambda_m]} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h},$$

$$S_m^0 = \frac{1}{\pi} \sin(\pi\lambda_m^0) \sum_{h=-[\eta(2M+1)]}^{[\eta(2M+1)]-1} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h},$$

where $0 < \eta < \min(1, 1/y_1)$. Our aim is to show that the sums S_m^-, S_m^+ and $|S_m^0 - 1|$ are $O(1/M)$ with constants depending on η but not on m. Note, however, that the possibility of such a decomposition imposes restrictions on m and η. In particular, the sum S_m^0 has $2[\eta(2M+1)]$ terms (as we shall require) only if

$$-[\eta(2M+1)] - [\lambda_m] \geqslant -(2M+1) \tag{7}$$

which is certainly satisfied if

$$m \leqslant (1-\eta)\, M_c - \frac{\eta}{2}. \tag{8}$$

If $c > 1$ and $\eta < 1 - 1/c$ then (8) and consequently also (7) is satisfied for all $m \leqslant M$.

Consider first the case $y_1 = 0$. Then by the partial fraction decomposition of $\pi/\sin \pi\lambda_m^0$

$$\begin{aligned} S_m(-(2M+1), 2M+1) &= \frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=-(2M+1)+[\lambda_m]}^{2M+1+[\lambda_m]} \frac{(-1)^h}{\lambda_m^0 + h} \\ &= \frac{\sin \pi\lambda_m^0}{\pi} \left(\frac{\pi}{\sin \pi\lambda_m^0} + \frac{2\theta}{2M+1-[\lambda_m]-\lambda_m^0+1} \right), \end{aligned}$$

where $0 \leqslant |\theta| \leqslant 1$ and therefore by (7)

$$|S_m(-(2M+1), 2M+1) - 1| \leqslant \frac{2}{\pi[\eta(2M+1)]} \leqslant \frac{2}{\pi\eta M} \qquad \text{if} \quad M > \frac{1}{\eta}.$$

Let now $y_1 > 0$. In what follows, for $\alpha > 0$ we denote by w_α the continuity module of the function $\cos t/t$ in the interval $[\alpha, \infty]$, i.e.,

$$w_\alpha(\delta) = \sup \left\{ \left| \frac{\cos t_1}{t_1} - \frac{\cos t_2}{t_2} \right| : \alpha \leqslant t_1 < t_2 \leqslant t_1 + \delta \right\}.$$

We start with

$$\begin{aligned} \frac{\pi}{\sin(\pi\lambda_m^0)} S_m^+ = \frac{1}{2} \sum_{\substack{h \geqslant [\eta(2M+1)] \\ h \equiv 0 (\mathrm{mod}\ 2)}}^{2M+1+[\lambda_m]} \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\pi \dfrac{\lambda_m^0 + h}{2M+1} y_1} \frac{2\pi y_1}{2M+1} \\ - \frac{1}{2} \sum_{\substack{h \geqslant [\eta(2M+1)] \\ h \equiv 1 (\mathrm{mod}\ 2)}}^{2M+1+[\lambda_m]} \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\pi \dfrac{\lambda_m^0 + h}{2M+1} y_1} \frac{2\pi y_1}{2M+1} \end{aligned}$$

For $\eta \in \,]0, 1[\, \cap \,]0, 1/y_1[$ and $M > 1/\eta$ we have

$$\frac{\eta}{2} < \frac{\tau_1}{\pi y_1} = \frac{\lambda_m^0 + [\eta(2M+1)]}{2M+1} \leqslant \frac{\lambda_m^0 + h}{2M+1} \leqslant \frac{\tau_2}{\pi y_1} = \frac{\lambda_m + 2M+1}{2M+1} \leqslant \frac{1+c}{c}$$

and therefore by the Riemann sum lemma, putting $\alpha = (\pi\eta y_1)/2$,

$$\left| \sum_{\substack{h \geqslant [\eta(2M+1)] \\ h \equiv i (\mathrm{mod}\ 2)}}^{2M+1+[\lambda_m]} \frac{\cos \pi \frac{\lambda_m^0 + h}{2M+1} y_1}{\pi \frac{\lambda_m^0 + h}{2M+1} y_1} \frac{2\pi y_1}{2M+1} - \int_{\tau_1}^{\tau_2} \frac{\cos t}{t} dt \right|$$
$$< w_\alpha \left(\frac{2\pi y_1}{2M+1} \right) \frac{1+c}{c} \pi y_1 + \frac{4}{\eta(2M+1)}.$$

Applying this estimate for $i = 0$ as well as for $i = 1$ we obtain

$$|S_m^+| < w_\alpha \left(\frac{2\pi y_1}{2M+1} \right) \frac{1+c}{c} y_1 + \frac{4}{\pi\eta(2M+1)}. \tag{9}$$

If the sum S_m^- is non-void at all, then applying a similar reasoning as above we obtain

$$|S_m^-| < w_\alpha \left(\frac{2\pi y_1}{2M+1} \right) y_1 + \frac{4}{\pi\eta(2M+1)}.$$

Finally, under the assumption (8), the sum S_m^0 may be decomposed as follows, using the partial fraction decomposition of $\pi/\sin \pi\lambda_m^0$,

$$S_m^0 = \frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=-[\eta(2M+1)]}^{[\eta(2M+1)]-1} \frac{(-1)^h}{\lambda_m^0 + h}$$
$$- \frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=-[\eta(2M+1)]}^{[\eta(2M+1)]-1} (-1)^h \frac{1 - \cos \pi \frac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h}$$
$$= \frac{\sin \pi\lambda_m^0}{\pi} \left(\frac{\pi}{\sin \pi\lambda_m^0} + \frac{2\theta}{[\eta(2M+1)]} \right)$$
$$- \frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=-[\eta(2M+1)]}^{[\eta(2M+1)]-1} (-1)^h \frac{1 - \cos \pi \frac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h},$$

where $|\theta| \leqslant 1$. As to the second sum observe that the function $g(t) = (1 - \cos t)/t$ $(g(0) = 0)$ is increasing for $0 \leqslant t \leqslant \pi/2$. Therefore, if η is chosen smaller than $1/2y_1$ then for $M > 1/\eta$ the second sum is absolutely smaller than $2/(\eta(2M+1))$. Consequently we have

$$|S_m^0 - 1| \leqslant \frac{4}{\pi\eta(2M+1)}.$$

These estimates may be summed up to an estimate of $S_m(-(2M+1), 2M+1)$ as follows:

LEMMA 2. *Suppose* $(2m+1)/c \notin \mathbb{Z}$ *and let*

$$0 < \eta < \min\left(1, \frac{1}{2y_1}\right),$$

$$\alpha = \frac{\pi \eta y_1}{2},$$

$$m \leqslant (1-\eta)\, M_c - \frac{\eta}{2},$$

$$M > \frac{1}{\eta}.$$

Then one has

$$|S_m(-(2M+1), 2M+1) - 1| \leqslant B_1(\eta, M) = \frac{1+2c}{c}\, y_1 w_\alpha \left(\frac{2\pi y_1}{2M+1}\right) + \frac{4}{\pi \eta M}.$$

The requirement on m *is automatically satisfied for all* $m \leqslant M$ *if* $c > 1$ *and* $\eta < 1 - 1/c$.

The estimate in Lemma 2 becomes ineffective for $\eta \to 0$, e.g., in the case $c \leqslant 1$ for values of m in a bounded neighbourhood of M_c. We proceed to show:

LEMMA 3. *For* $-(2M+1) \leqslant L_1 \leqslant L_2 \leqslant 2M+1$ *and* $(2m+1)/c \notin \mathbb{Z}$ *the sum* $S_m(L_1, L_2)$ *is bounded uniformly in* M *and* m.

Proof. A second look at (6) reveals that it is sufficient to exhibit positive constants C_1 and C_2 not depending on m and M with the property that

$$\left| \frac{\sin \pi \lambda_m^0}{\pi} \sum_{h=0}^{H} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1}\, y_1}{\lambda_m^0 + h} \right| \leqslant C_1 \qquad \text{as long as} \quad 0 < \frac{\lambda_m^0 + H}{2M+1}\, y_1 \leqslant \frac{1}{2} \tag{10}$$

(equivalently $H \leqslant \bar{H}$ where $\bar{H}$ alternatively stands for $[(2M+1)/2y_1]$ or $[(2M+1)/2y_1]-1$), and

$$\left|\frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=\bar{H}}^{H} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0+h}{2M+1}\, y_1}{\lambda_m^0+h}\right|$$
$$\leqslant C_2 \qquad \text{as long as} \quad \bar{H} < H \leqslant (2M+1)\left(1+\frac{1}{c}\right). \tag{11}$$

Note that (10) already implies

$$\left|\frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=H}^{\bar{H}} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0+h}{2M+1}\, y_1}{\lambda_m^0+h}\right| \leqslant 2C_1 \qquad \text{as long as} \quad 0 < H \leqslant \bar{H}.$$

As to the proof of (10), for $H=0$ we have

$$\left|\frac{\sin \pi\lambda_m^0}{\pi} \frac{\cos \pi \dfrac{\lambda_m^0}{2M+1}\, y_1}{\lambda_m^0}\right| \leqslant 1.$$

If $y_1 = 0$ this furnishes

$$\left|\frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=0}^{H} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0+h}{2M+1}\, y_1}{\lambda_m^0+h}\right| \leqslant 2 \qquad \text{for all} \quad H \geqslant 1,$$

which takes care of both (10) and (11).

If $y_1 > 0$, then for $1 \leqslant H \leqslant \bar{H}$ as in the decomposition of S_m^0 in the proof of Lemma 2 we obtain

$$\left|\frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=0}^{H} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0+h}{2M+1}\, y_1}{\lambda_m^0+h}\right|$$
$$\leqslant 1 + \left|\frac{1}{\pi} \sum_{h=1}^{H} \frac{(-1)^h}{\lambda_m^0+h}\right| + \left|\frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=1}^{H} (-1)^h \frac{1-\cos \pi \dfrac{\lambda_m^0+h}{2M+1}\, y_1}{\lambda_m^0+h}\right|$$
$$\leqslant 1 + \frac{1}{\pi} + \frac{1}{\pi H} \leqslant 1 + \frac{2}{\pi} = C_1.$$

The estimate (11) is furnished again by applying the Riemann sum lemma as in the estimate (9) of S_m^+ in the proof of Lemma 2, putting $\tau_1 = \pi/2$ and using that $\tau_2 \leqslant (1+1/c)\,\pi y_1$:

$$\left| \frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=\bar{H}}^{H} (-1)^h \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1}\, y_1}{\lambda_m^0 + h} \right|$$

$$\leqslant \left(y_1 \left(1 + \frac{1}{c} \right) - \frac{1}{2} \right) w_{\pi/2} \left(\frac{2\pi y_1}{3} \right) + \frac{4 y_1}{3\pi} = C_2. \quad \blacksquare$$

Proof of Assertion (b). We consider first under which conditions it may happen that for some $m \in [0, M]$ and some $l \in [-(2M+1), 2M+1]$ we have $(2m+1)/c = -l \in \mathbb{Z}$. In this case we necessarily have $c = a/b$ with relatively prime natural numbers a, b satisfying

$$\frac{(2m+1)\, b}{a} \in \mathbb{Z}$$

and

$$0 < \frac{(2m+1)\, b}{a} \leqslant 2M+1.$$

This is the case iff

$$2m+1 = (2k+1)\, a$$

and

$$0 \leqslant k \leqslant K_1 = \left[\frac{2M+1}{2b} - \frac{1}{2} \right].$$

Furthermore, the inequality

$$m = \frac{(2k+1)\, a - 1}{2} \leqslant M$$

is equivalent to

$$k \leqslant K_2 = \left[\frac{2M+1}{2a} - \frac{1}{2} \right].$$

For $K=\min(K_1, K_2)$ (in case $c=a/b\leqslant 1$ we have $K=K_1$) we obtain

$$\lim_{M\to\infty}\frac{2K+1}{2M+1}=\min\left(\frac{1}{b},\frac{1}{a}\right).$$

Therefore, if $c=a/b$, $(a, b)=1$, $a\equiv 1(\text{mod } 2)$, then (cf. (5))

$$s_M^{(2)}(x_1, c)=\frac{1}{2\alpha\pi}\sum_{k=0}^{K}\frac{\sin\left(\pi\dfrac{(2k+1)\,a}{2M+1}x_1\right)}{\pi\dfrac{(2k+1)\,a}{2M+1}x_1}\frac{2a\pi x_1}{2M+1}$$

$$\underset{M\to\infty}{\to}\frac{1}{2a\pi}\int_0^{\pi x_1\bar{c}}\frac{\sin s}{s}\,ds.$$

As to $s_M^{(3)}$, given $\eta\in\,]0, 1[$ as in Lemma 2 we choose M so large that in Lemma 2 we get $B_1(\eta, M)<\varepsilon$. We then decompose $s_M^{(3)}$ according to the following (possibly empty) ranges of the index m:

$$\Sigma_1\colon m<M^-=\min\left(M, (1-\eta)\,M_c-\frac{\eta}{2}\right),$$

$$\Sigma_2\colon M^-\leqslant m\leqslant M^+=\min\left(M, (1+\eta)\,M_c-\frac{\eta}{2}\right),$$

$$\Sigma_3\colon M^+<m.$$

In the sequel it will be convenient to admit non-integers as summation limits as long as the meaning will be clear from the context; e.g., we shall write $\sum_{m>\alpha}^{\beta}$ for $\sum_{m=[\alpha]+1}^{[\beta]}$.

If c is irrational or $c=a/b$, $(a, b)=1$, $a\equiv 0$ (mod 2), then

$$\sum\nolimits_1=\frac{1}{2\pi}\sum_{m<M^-}\frac{\sin\pi\dfrac{2m+1}{2M+1}x_1}{\pi\dfrac{2m+1}{2M+1}x_1}\frac{2\pi x_1}{2M+1}$$

$$+\frac{1}{2\pi}\sum_{m<M^-}\frac{\sin\pi\dfrac{2m+1}{2M+1}x_1}{\pi\dfrac{2m+1}{2M+1}x_1}\frac{2\pi x_1}{2M+1}(S_m(-(2M+1), 2M+1)-1). \tag{12}$$

Let $\gamma = \min(1, c(1-\eta))$. Our standard Riemann-sum argument together with Lemma 2 shows that

$$\left|\sum\nolimits_1 - \frac{1}{2\pi}\int_0^{\gamma\pi x_1} \frac{\sin s}{s}\, ds\right|$$

becomes smaller than $\varepsilon/2\pi\ |\int_0^{\gamma\pi x_1} (\sin s/s)\, ds|$ if M is sufficiently large. It remains to choose η sufficiently small to begin with.

On the other hand, if $c = a/b$, $(a, b) = 1$, $a \equiv 1 \pmod 2$, then for $0 \leqslant m \leqslant M$ the integer $(2m+1)\, b$ runs periodically through the residue classes modulo a. In (12) all terms corresponding to numerators $2m+1$ divisible by a have to be replaced by 0 with the consequence that now

$$\left|\sum_1 - \frac{a-1}{2a\pi}\int_0^{\gamma\pi x_1} \frac{\sin s}{s}\, ds\right|$$

becomes smaller than $(a-1)\, \varepsilon/a\pi\ |\int_0^{\gamma\pi x_1} (\sin s/s)\, ds|$.

If B is a bound for $S_m(-(2M+1), 2M+1)$ uniformly in m and M (Lemma 3), then

$$\left|\sum\nolimits_2\right| \leqslant \frac{2\eta M_c}{\pi(1-\eta)\, c(2M+1)}\, B \leqslant \frac{\eta}{\pi(1-\eta)}\, B$$

which can be made arbitrarily small by a suitable choice of η.

Finally, if $\sum_3$ is not void, then for $m > M^+$ and $2M+1 > 2/c$ we have

$$\lambda_m^0 + h = \lambda_m + l \geqslant \eta(2M+1) - \frac{2\eta}{c} > 0.$$

If $y_1 = 0$ then

$$|S_m(-(2M+1), 2M+1)| = \left|\frac{\sin \pi\lambda_m^0}{\pi} \sum_{h=[\lambda_m]-(2M+1)}^{2M+1+[\lambda_m]} \frac{(-1)^h}{\lambda_m^0 + h}\right|$$
$$\leqslant \frac{1}{\pi}\, \frac{c}{c\eta(2M+1) - 2\eta} = O\left(\frac{1}{M}\right).$$

If $y_1 > 0$ and, e.g., $\eta < c/4$ then $S_m(-(2M+1), 2M+1)$ coincides with S_m^+ in the proof of Lemma 2 up to a single term absolutely not larger than $1/(\eta(2M+1) - 2)$. By the estimate (9) every corresponding sum $S_m(-(2M+1), 2M+1)$ becomes uniformly small for $M \to \infty$ and so does $\sum_3$. This concludes the proof of assertion (b). ∎

For the proofs of assertions (c) and (d) let v be the continuity module of the function $\sin t/t$, taken over the entire real line. Observe that $v(\delta) \leqslant \delta$.

Proof of Assertion (c). In $s_M^{(4)}$ we need only to consider the case $y_1 > 0$ and integers m for which $\lambda_m^0 \neq 0$. Similarly as in the proof of Lemma 2 by the Riemann sum lemma we obtain

$$\left| \sum_{l=-(2M+1)}^{2M+1} \sin \pi(\lambda_m + l) \frac{\sin \pi \dfrac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} \right|$$
$$= \left| \sin \pi\lambda_m \sum_{l=-(2M+1)}^{2M+1} (-1)^l \frac{\sin \pi \dfrac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} \right|$$
$$\leqslant 2v\left(\frac{2\pi y_1}{2M+1}\right) \pi y_1 + \frac{1}{M}.$$

We now have to deal with the fact that the function $\cos t/t$ is not Riemann-integrable on an interval containing 0. For $x_1 > 0$ and $\eta \in]0, 1[$ we use the decomposition

$$s_M^{(4)} = \sum_{m < \eta M} + \sum_{m \geqslant \eta M}.$$

This leads to

$$|s_M^{(4)}| \leqslant \frac{1}{\pi^2}\left(2v\left(\frac{2\pi y_1}{2M+1}\right)\pi y_1 + \frac{1}{M}\right)\eta M$$
$$+ \frac{1}{2\pi^2} \sum_{m \geqslant \eta M} \left| \frac{\cos \pi \dfrac{2m+1}{2M+1} x_1}{\pi \dfrac{2m+1}{2M+1} x_1} \frac{2\pi x_1}{2M+1} \right| \left(2v\left(\frac{2\pi y_1}{2M+1}\right)\pi y_1 + \frac{1}{M}\right).$$

The first term may be made arbitrarily small by a suitable choice of η while the second term converges for $M \to \infty$ to $\int_{\eta\pi x_1}^{\pi x_1} (|\cos t|/t)\, dt$ times 0. For $x_1 = 0$ we obtain

$$\sum_{m=0}^{M} \frac{1}{2m+1} \left| \sum_{l=-(2M+1)}^{2M+1} \sin \pi(\lambda_m + l) \frac{\sin \pi \dfrac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} \right|$$
$$\leqslant \left(1 + \frac{1}{2}\log(2M+1)\right) \cdot O\left(\frac{1}{M}\right)$$
$$= o(1) \qquad \text{as} \quad M \to \infty. \quad \blacksquare$$

Proof of Assertion (d). We have

$$s_M^{(5)} = \frac{2}{\pi^2} \sum_{m=0}^{M} \frac{\sin \pi \dfrac{2m+1}{2M+1} x_1}{2m+1} \cdot \left\{ \sin^2 \frac{\pi \lambda_m}{2} \sum_{\substack{l \geqslant -(2M+1) \\ l \equiv 0 (\mathrm{mod}\ 2)}}^{2M+1} \frac{\sin \pi \dfrac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} \right.$$

$$\left. + \cos^2 \frac{\pi \lambda_m}{2} \sum_{\substack{l \geqslant -(2M+1) \\ l \equiv 1 (\mathrm{mod}\ 2)}}^{2M+1} \frac{\sin \pi \dfrac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} \right\}, \tag{13}$$

where one of the terms in curly brackets is replaced by zero if λ_m is an even resp. odd integer.

Our standard Riemann sum argument furnishes

$$\left| \sum_{\substack{l \geqslant -(2M+1) \\ l \equiv i (\mathrm{mod}\ 2)}}^{2M+1} \frac{\sin \pi \dfrac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} - \frac{1}{2} \int_{\tau_1}^{\tau_2} \frac{\sin s}{s}\, ds \right| \leqslant \frac{1}{2} \left(2v \left(\frac{2\pi y_1}{2M+1} \right) \pi y_1 + \frac{1}{M} \right), \tag{14}$$

where

$$\tau_1 = \pi \frac{\lambda_m - (2M+1)}{2M+1} y_1 = \pi y_1 \left(\frac{2m+1}{c(2M+1)} - 1 \right),$$

$$\tau_2 = \pi \frac{\lambda_m + 2M+1}{2M+1} y_1 = \pi y_1 \left(\frac{2m+1}{c(2M+1)} + 1 \right).$$

Let us introduce the notation

$$g(t) = \int_{(t y_1 / c x_1) - \pi y_1}^{(t y_1 / c x_1) + \pi y_1} \frac{\sin s}{s}\, ds.$$

By (14) we have

$$\left| \sum_{\substack{l \geqslant -(2M+1) \\ l \equiv i (\mathrm{mod}\ 2)}}^{2M+1} \frac{\sin \pi \dfrac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} - \frac{1}{2}\, g\left(\pi \frac{2m+1}{2M+1} x_1 \right) \right|$$

$$\leqslant \frac{1}{2} \left(2v \left(\frac{2\pi y_1}{2M+1} \right) \pi y_1 + \frac{1}{M} \right).$$

This estimate combined with (13) gives

$$s_M^{(5)} = \frac{1}{2\pi^2} \sum_{m=0}^{M} \frac{\sin \pi \dfrac{2m+1}{2M+1} x_1}{\pi \dfrac{2m+1}{2M+1} x_1} \, g\left(\pi \frac{2m+1}{2M+1} x_1\right) \frac{2\pi x_1}{2M+1} + o(1)$$

$$\text{as} \quad M \to \infty.$$

This amounts to assertion (d). ▮

In the proof of assertion (e), as in the proof of assertion (c), we have to deal with the fact that the function $\cos t/t$ is not Riemann-integrable on an interval containing 0. In order to appreciate the following considerations a picture might be helpful. Let us associate with every summand of the multiple sum

$$s_M^{(6)}(x_1, y_1, c)$$

$$= -\frac{2}{\pi^2} \sum_{m=0}^{M} \frac{\cos \pi \dfrac{2m+1}{2M+1} x_1}{2m+1}$$

$$\times \sum_{\substack{l=-(2M+1) \\ l \neq -(2m+1)/c}}^{2M+1} \sin^2 \frac{\pi}{2}\left(\frac{2m+1}{c} + l\right) \frac{\cos \pi \dfrac{(2m+1)/c + l}{2M+1} y_1}{\dfrac{2m+1}{c} + l}$$

$$= -\frac{2}{\pi^2} \sum_{m=0}^{M} \frac{\cos \pi \dfrac{2m+1}{2M+1} x_1}{2m+1}$$

$$\times \sum_{\substack{h=[\lambda_m]-(2M+1) \\ \lambda_0^m + h \neq 0}}^{[\lambda_m]+2M+1} \sin^2 \frac{\pi}{2} (\lambda_m^0 + h) \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h}$$

a point $P(m, h) = (\mu = (2m+1)/(2M+1),\ \nu = h/(2M+1)) \in \mathbb{R}^2$. In $s_M^{(6)}$ the summation essentially is executed over all points $P(m, h) = (\mu, \nu)$ in the domain

$$G \cdots 0 \leqslant \mu \leqslant 1, \qquad -1 + \frac{\mu}{c} \leqslant \nu \leqslant 1 + \frac{\mu}{c}. \tag{15}$$

Consider the following subdomains:

$$G_1 \cdots 0 \leqslant \mu \leqslant \eta, \quad 1 - \frac{\mu}{c} \leqslant v \leqslant 1 + \frac{\mu}{c}; \tag{16}$$

$$G_2 \cdots 0 \leqslant \mu \leqslant \bar{c} = \min(c, 1), \quad \frac{\mu}{c} - 1 \leqslant v \leqslant 1 - \frac{\mu}{c}; \tag{17}$$

$$G_3 \cdots (1-\eta)\, c \leqslant \mu \leqslant \min(1, 1+\eta)\, c), \quad \left|1 - \frac{\mu}{c}\right| \leqslant v \leqslant \eta. \tag{18}$$

(If $c > 1$ then, for sufficiently small η, the subdomain G_3 is empty.) We shall show that, for large M, the sums over each of the domains G_1, G_2, G_3 may be made small (by a suitable choice of η) while the sum over the remaining points comes close to the limit indicated in assertion (e). Since these domains serve only to illustrate what is going on we shall accept slight discrepancies between the corresponding sets of lattice points $P(m, h)$ and the actual sums to be estimated, due to the influence of the term λ_m^0.

Lemma 4 gives an estimate for the sum over the points in G_1.

LEMMA 4. *Let $\eta \in \,]0, 1[\, \cap \,]0, c[$. Then*

$$\sum_{m=0}^{\eta M} \frac{1}{2m+1} \sum_{l=2M+1-2[\lambda_m]}^{2M+1} \frac{1}{\lambda_m + l} \leqslant \frac{\eta(2+c)}{2(c-\eta)} + O\left(\frac{1}{M}\right) \qquad \textit{as} \quad M \to \infty.$$

Proof.

$$\begin{aligned} &\sum_{m=0}^{\eta M} \frac{1}{2m+1} \sum_{l=2M+1-2[\lambda_m]}^{2M+1} \frac{1}{\lambda_m + l} \\ &\qquad \leqslant \sum_{m=0}^{\eta M} \frac{1}{2m+1} \cdot \frac{2\lambda_m + 1}{2M+1-\lambda_m} \\ &\qquad \leqslant (\eta M + 1) \frac{2+c}{2M(c-\eta)-1} \\ &\qquad = \frac{\eta(2+c)}{2(c-\eta)} + O\left(\frac{1}{M}\right) \qquad \text{as} \quad M \to \infty. \quad \blacksquare \end{aligned}$$

Lemma 5 gives an estimate for the sum over the points in G_3.

Lemma 5. *If* $c \leqslant 1$ *then for* $\eta \in]0, 1[$ *and as* $M \to \infty$ *one has*

$$s^{+}(\eta, M) = \sum_{m > M_c}^{\min(M, (1+\eta) M_c)} \frac{1}{2m+1} \sum_{\substack{l=(2M+1) \\ \lambda_m + l \neq 0}}^{\eta(2M+1)-\lambda_m} \frac{\sin^2 \frac{\pi}{2}(\lambda_m + l)}{\lambda_m + l}$$

$$\leqslant \frac{\eta(c+1)}{c} + O\left(\frac{1}{M}\right),$$

$$s^{-}(\eta, M) = \sum_{m \geqslant (1-\eta) M_c}^{M_c} \frac{1}{2m+1} \sum_{\substack{l \geqslant 2M+1-2\lambda_m \\ \lambda_m + l \neq 0}}^{\eta(2M+1)-\lambda_m} \frac{\sin^2 \frac{\pi}{2}(\lambda_m + l)}{\lambda_m + l}$$

$$\leqslant \frac{\eta(c+3)}{2c(1-\eta)} + O\left(\frac{1}{M}\right).$$

Proof. We shall prove the estimate for $s^{-}(\eta, M)$; the first assertion may be proved similarly. Obviously the inequalities $m \leqslant M_c$ and $0 \leqslant 2M + 1 - \lambda_m$ are equivalent. For $2M + 1 - 2\lambda_m \leqslant l \leqslant \eta(2M+1) - \lambda_m$ we therefore have

$$0 \leqslant 2M + 1 - \lambda_m \leqslant \lambda_m + l = \lambda_m^0 + h \leqslant \eta(2M+1). \tag{19}$$

The term with the smallest denominator in the second sum is $\sin^2((\pi/2)\lambda_m^0)/\lambda_m^0 \leqslant \pi/2$ for $\lambda_m^0 > 0$ (and $h = 0$), and 1 for $\lambda_m^0 = 0$ (and $h = 1$) otherwise. Counting how many values of m admit the value $h = 0$ we find that

$$0 \leqslant 2M + 1 - \lambda_m < 1$$

is equivalent with

$$cM - \frac{1}{2} < m \leqslant cM + \frac{c-1}{2}$$

which is satisfied for at most $[c/2] + 1$ values of m. Furthermore, the inequality $m \geqslant (1-\eta) M_c$ implies

$$2m + 1 \geqslant 2(1-\eta)(cM - 1). \tag{20}$$

This furnishes the estimate

$$s^-(\eta, M) \leqslant \frac{\pi(c+2)}{8(1-\eta)(cM-1)} + \sum_{m \geqslant (1-\eta) M_c}^{M_c} \frac{1}{2m+1} \sum_{h \geqslant \max(1, 2M+1-\lambda_m-\lambda_m^0)}^{\eta(2M+1)-\lambda_m^0} \frac{1}{\lambda_m^0+h}.$$

Given $h \geqslant 1$, the middle inequality of (19) implies

$$M_c - \frac{c}{2}(h+\lambda_m^0) \leqslant m.$$

Rearranging the summation we therefore obtain for $c \leqslant 1$

$$\begin{aligned} s^-(\eta, M) &\leqslant \sum_{h=1}^{\eta(2M+1)} \frac{1}{h} \sum_{m \geqslant M_c - c/2(h+1)}^{M_c} \frac{1}{2m+1} + O\left(\frac{1}{M}\right) \\ &\leqslant \sum_{h=1}^{\eta(2M+1)} \frac{1}{h} \cdot \frac{ch+3}{4(1-\eta)(Mc-1)} + O\left(\frac{1}{M}\right) \qquad \text{by (20)} \\ &\leqslant \eta(2M+1) \cdot \frac{c+3}{4(1-\eta)(Mc-1)} + O\left(\frac{1}{M}\right) \\ &= \frac{\eta(c+3)}{2c(1-\eta)} + O\left(\frac{1}{M}\right). \quad \blacksquare \end{aligned}$$

In order to obtain an estimate of the sum over the points in G_2 we now turn to a study of sums of the type

$$\sum_m = \sum_{\substack{l=-(2M+1) \\ \lambda_m+l \neq 0}}^{2M+1-2[\lambda_m]-1} \sin^2 \frac{\pi}{2}(\lambda_m+l) \frac{\cos \pi \dfrac{\lambda_m+l}{2M+1} y_1}{\lambda_m+l}$$

$$= \sum_{h=[\lambda_m]-(2M+1)}^{2M+1-[\lambda_m]-1} * \sin^2 \frac{\pi}{2}(\lambda_m^0+h) \frac{\cos \pi \dfrac{\lambda_m^0+h}{2M+1} y_1}{\lambda_m^0+h}.$$

The $*$ signals that for $\lambda_m \in \mathbb{Z}$, $l = -\lambda_m$ (equivalently for $\lambda_m^0 = h = 0$) the corresponding term is replaced by zero. These sums are characterized by the fact that they contain the same number of terms with nonnegative denominator (for non-negative values of h) as of terms with negative denominators (for negative values of h).

LEMMA 6. *For* $\eta \in]0, 1[\cap]0, (1/2y_1)[\cap]0, c[$ *and* $m \leqslant (1-\eta) M_c$ *one has* $\sum_m = O(1/M)$ *where the constant depends on* η *but not on* m.

Proof. Note that under the mentioned conditions the number of terms in $\sum_m$ exceeds $2\eta(2M+1-1/c)$. We assume M to be large enough so that this is positive. Furthermore, for sufficiently large M (the bound depending on η and y_1) the requirement $\eta < 1/2y_1$ guarantees $\pi((\lambda_m^0 + [\eta(2M+1)])/(2M+1))\, y_1 < \pi/2$ while the requirement $\eta < c$ guarantees $\eta(2M+1) - 1 < \eta(2M+1-1/c)$.

Suppose first that $\lambda_m^0 \neq 0$. We shall decompose sums of this type in the following way,

$$\sum\nolimits_m = \overset{-}{\sum}\nolimits_{m,\eta} + \overset{0}{\sum}\nolimits_{m,\eta} + \overset{+}{\sum}\nolimits_{m,\eta},$$

where

$$\overset{-}{\sum}\nolimits_{m,\eta} = \sum_{h=[\lambda_m]-(2M+1)}^{-[\eta(2M+1)]-1},$$

$$\overset{0}{\sum}\nolimits_{m,\eta} = \sum_{h=-[\eta(2M+1)]}^{[\eta(2M+1)]-1},$$

$$\overset{+}{\sum}\nolimits_{m,\eta} = \sum_{h=[\eta(2M+1)]}^{2M+1-[\lambda_m]-1}.$$

We have

$$\begin{aligned}\overset{-}{\sum}\nolimits_{m,\eta} &= \sin^2\left(\frac{\pi}{2}\lambda_m^0\right) \sum_{\substack{h \geqslant [\lambda_m]-(2M+1) \\ h \equiv 0 (\mathrm{mod}\ 2)}}^{-[\eta(2M+1)]-1} \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1}\, y_1}{\lambda_m^0 + h} \\ &\quad + \cos^2\left(\frac{\pi}{2}\lambda_m^0\right) \sum_{\substack{h \geqslant [\lambda_m]-(2M+1) \\ h \equiv 1 (\mathrm{mod}\ 2)}}^{-[\eta(2M+1)]-1} \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1}\, y_1}{\lambda_m^0 + h} \\ &= \sin^2\left(\frac{\pi}{2}\lambda_m^0\right) \overset{0-}{\sum}\nolimits_{m,\eta} + \cos^2\left(\frac{\pi}{2}\lambda_m^0\right) \overset{1-}{\sum}\nolimits_{m,\eta}.\end{aligned}$$

By the Riemann sum lemma and similarly as in the proof of assertion (c) for $i \in \{0, 1\}$ we get, putting $\alpha = \pi y_1 \eta/2$,

$$\left| \overset{i-}{\sum}\nolimits_{m,\eta} - \frac{1}{2} \int_{\tau_1}^{\tau_2} \frac{\cos t}{t}\, dt \right| \leqslant \frac{1}{2} \left(w_\alpha \left(\frac{2\pi y_1}{2M+1} \right) (\tau_2 - \tau_1) + \frac{\cos \alpha}{\alpha} \frac{2\pi y_1}{2M+1} \right),$$

where

$$\tau_1 = \pi \frac{\lambda_m - (2M+1)}{2M+1} y_1 = \frac{\pi}{c} \frac{2m+1}{2M+1} y_1 - \pi y_1,$$

$$\tau_2 = \pi \frac{\lambda_m^0 - \eta(2M+1) + \theta - 1}{2M+1} y_1 = -\pi y_1 \eta + \pi \frac{\lambda_m^0 + \theta - 1}{2M+1} y_1 \qquad (0 \leqslant \theta < 1).$$

This implies

$$\left| \sum_{m,\eta}^{-} - \frac{1}{2} \int_{(\pi/c)(2m+1)/(2M+1)\, y_1 - \pi y_1}^{-\pi y_1 \eta} \frac{\cos t}{t}\, dt \right| \leqslant \frac{1}{2} \left(w_\alpha \left(\frac{2\pi y_1}{2M+1} \right) \pi y_1 (1-\eta) + O\left(\frac{1}{M} \right) \right)$$

and by the similarly obtained estimate for $\sum_{m,\eta}^{+}$

$$\left| \sum_{m,\eta}^{+} - \frac{1}{2} \int_{\pi y_1 \eta}^{\pi y_1 - (\pi/c)(2m+1)/(2M+1)\, y_1} \frac{\cos t}{t}\, dt \right| \leqslant \frac{1}{2} \left(w_\alpha \left(\frac{2\pi y_1}{2M+1} \right) \pi y_1 (1-\eta) + O\left(\frac{1}{M} \right) \right)$$

finally

$$\left| \sum_{m,\eta}^{-} + \sum_{m,\eta}^{+} \right| \leqslant w_\alpha \left(\frac{2\pi y_1}{2M+1} \right) \pi y_1 (1-\eta) + O\left(\frac{1}{M} \right).$$

Since the function $\cos t/t$ has a bounded derivative on $[\alpha, \infty[$ the resulting estimate is $O(1/M)$ with a bound depending on η but not on m.

By the initial remark in the proof of Lemma 6, because of the condition $m \leqslant M_c$ the sum $\sum_{m,\eta}^{0}$ contains indeed $2[\eta(2M+1)]$ terms. Note that for $c > 1$ and $\eta < 1 - 1/c$ we have $m \leqslant (1-\eta) M_c$ for all $m \leqslant M$. For an estimate of $\sum_{m,\eta}^{0}$ we write

$$\begin{aligned} \sum_{m,\eta}^{0} &= \sum_{h=-[\eta(2M+1)]}^{[\eta(2M+1)]-1} \sin^2 \left(\frac{\pi}{2} (\lambda_m^0 + h) \right) \frac{1}{\lambda_m^0 + h} \\ &\quad - \sum_{h=-[\eta(2M+1)]}^{[\eta(2M+1)]-1} \sin^2 \left(\frac{\pi}{2} (\lambda_m^0 + h) \right) \frac{1 - \cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h} \\ &= \sum_{m,\eta}^{0\,\prime} - \sum_{m,\eta}^{0\,\prime\prime}. \end{aligned}$$

As to the first sum, using the partial fraction decompositions of $\pi/\sin \pi x$ and of $\pi/2 \tan(\pi/2)\, x$ we obtain

$$\begin{aligned}
\sum\nolimits_{m,\eta}^{0\prime} &= \sin^2 \frac{\pi}{2}\lambda_m^0 \sum_{-[\eta(2M+1)] \leqslant 2h < [\eta(2M+1)]} \frac{1}{\lambda_m^0 + 2h} \\
&\quad + \cos^2 \frac{\pi}{2}\lambda_m^0 \sum_{-[\eta(2M+1)] \leqslant 2h-1 < [\eta(2M+1)]} \frac{1}{\lambda_m^0 + 2h - 1} \\
&= \sin^2 \frac{\pi}{2}\lambda_m^0 \sum_{h=-[\eta(2M+1)]}^{[\eta(2M+1)]-1} \frac{(-1)^h}{\lambda_m^0 + h} \\
&\quad + \sum_{-[\eta(2M+1)] \leqslant 2h-1 < [\eta(2M+1)]} \frac{1}{\lambda_m^0 + 2h - 1} \\
&= \sin^2 \frac{\pi}{2}\lambda_m^0 \left(\frac{\pi}{\sin \pi\lambda_m^0} + O_1\left(\frac{1}{M}\right)\right) + \left(-\frac{\pi}{2}\tan\frac{\pi}{2}\lambda_m^0 + O_2\left(\frac{1}{M}\right)\right) \\
&= \frac{\pi}{2}\,\frac{\sin\frac{\pi}{2}\lambda_m^0}{\cos\frac{\pi}{2}\lambda_m^0} + O_1\left(\frac{1}{M}\right)\sin^2\frac{\pi}{2}\lambda_m^0 - \frac{\pi}{2}\tan\frac{\pi}{2}\lambda_m^0 + O_2\left(\frac{1}{M}\right) \\
&= O\left(\frac{1}{M}\right),
\end{aligned}$$

where the constants in $O_1(1/M)$, $O_2(1/M)$, and $O(1/M)$ depend on η but not on m.

Recall that we require $\eta < 1/2y_1$ which guarantees $|\pi((\lambda_m^0 + h)/(2M+1))\, y_1| < \pi/2$ for $-[\eta(2M+1)] \leqslant h < [\eta(2M+1)]$ and for M sufficiently large, independently of m. Writing

$$\begin{aligned}
\sum\nolimits_{m,\eta}^{0\prime\prime} &= \sin^2 \frac{\pi}{2}\lambda_m^0 \sum_{-[\eta(2M+1)] \leqslant 2h < [\eta(2M+1)]} \frac{1 - \cos \pi \dfrac{\lambda_m^0 + 2h}{2M+1}\, y_1}{\lambda_m^0 + 2h} \\
&\quad + \cos^2 \frac{\pi}{2}\lambda_m^0 \sum_{-[\eta(2M+1)] \leqslant 2h-1 < [\eta(2M+1)]} \frac{1 - \cos \pi \dfrac{\lambda_m^0 + 2h}{2M+1}\, y_1}{\lambda_m^0 + 2h - 1}
\end{aligned}$$

we observe again that the function $(1 - \cos t)/t$ is increasing for $0 \leqslant t \leqslant \pi/2$. Therefore in both sums the absolute values of the terms are increasing for increasing $|h|$. Consequently the absolute value of either sum is not greater than $2y_1/(2M+1)$. The same is therefore also true for $\sum_{m,\eta}^{0\prime\prime}$. This completes the proof of Lemma 6 in case $\lambda_m^0 \neq 0$.

If $\lambda_m^0 = 0$ then $\lambda_m \in \mathbb{Z}$. For $m \leqslant (1-\eta)\, M_c$ we obtain

$$2M+1-\lambda_m \geqslant \eta\left(2M+1-\frac{1}{c}\right) \geqslant \eta(2M+1)-1$$

and therefore again

$$\left|\sum\nolimits_m\right| \leqslant \left|\frac{1}{2M+1-\lambda_m}\right| \leqslant \frac{1}{\eta(2M+1)-1} < \frac{1}{\eta M}$$

for sufficiently large M, the bound depending on η but not on m. ▮

Without any restriction on any m we may use the following fact.

LEMMA 7. *For* $0<|\theta|<1$, $H \in \mathbb{N}$, *and* $|\pi((\theta+2H)/(2M+1))\, y_1| \leqslant \pi/2$ *one has*

$$\left|\sin^2\frac{\pi}{2}\theta \sum_{|h| \leqslant H} \frac{\cos \pi \dfrac{\theta+2h}{2M+1}\, y_1}{\theta+2h}\right| < 3.$$

Proof. For $h=0$ we have $\sin^2(\pi/2)\,\theta/\theta \leqslant \pi/2$. For $1 \leqslant |h| \leqslant H$ the sum may be arranged into an alternating sum with absolutely decreasing terms. This implies the assertion. ▮

LEMMA 8. *If* $H \in \mathbb{N}$ *and* $|\pi((\lambda_m^0+H)/(2M+1))\, y_1| \leqslant \pi/2$ *then*

$$\left|\sum_{h=-H}^{H-1} {}^{*} \sin^2\frac{\pi}{2}(\lambda_m^0+h) \frac{\cos \pi \dfrac{\lambda_m^0+h}{2M+1}\, y_1}{\lambda_m^0+h}\right| \leqslant 6.$$

Proof. For $\lambda_m^0 = 0$ we have

$$\sum_{h=-H}^{H} {}^{*} \sin^2\left(\frac{\pi}{2}h\right) \frac{\cos \pi \dfrac{h}{2M+1}\, y_1}{h} = 0$$

and therefore

$$\left|\sum_{h=-H}^{H-1} \sin^2\left(\frac{\pi}{2}h\right) \frac{\cos \pi \dfrac{h}{2M+1}\, y_1}{h}\right| \leqslant \frac{1}{H}.$$

For $\lambda_m^0 \neq 0$ apply Lemma 7 to both sums

$$\sin^2 \frac{\pi}{2} \lambda_m^0 \sum_{-H \leqslant 2h \leqslant H-1} \frac{\cos \pi \dfrac{\lambda_m^0 - 1 + 2h}{2M+1} y_1}{\lambda_m^0 - 1 + 2h},$$

$$\cos^2 \frac{\pi}{2} \lambda_m^0 \sum_{-H+1 \leqslant 2h \leqslant H} \frac{\cos \pi \dfrac{\lambda_m^0 - 1 + 2h}{2M+1} y_1}{\lambda_m^0 - 1 + 2h}$$

$$= \sin^2 \frac{\pi}{2} (\lambda_m^0 - 1) \sum_{-H \leqslant 2h \leqslant H} \frac{\cos \pi \dfrac{\lambda_m^0 - 1 + 2h}{2M+1} y_1}{\lambda_m^0 - 1 + 2h}. \quad \blacksquare$$

LEMMA 9. *Let* $0 < H \leqslant (1 + 1/c)(2M+1)$. *Then*

$$\left| \sum_{h=-H}^{H-1} {}^* \sin^2 \frac{\pi}{2} (\lambda_m^0 + h) \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h} \right| \leqslant 6 + O\left(\frac{1}{M}\right),$$

the constant not depending on m.

Proof. For $y_1 = 0$ and $\lambda_m \neq 0$ we have

$$\sum_{h=-H}^{H-1} \sin^2 \frac{\pi}{2} (\lambda_m^0 + h) \frac{1}{\lambda_m^0 + h}$$

$$= \sin^2 \frac{\pi}{2} \lambda_m^0 \sum_{\substack{h \geqslant -H \\ h \equiv 0 (\mathrm{mod}\ 2)}}^{H-1} \frac{1}{\lambda_m^0 + h} + \cos^2 \frac{\pi}{2} \lambda_m^0 \sum_{\substack{h \geqslant -H \\ h \equiv 1 (\mathrm{mod}\ 2)}}^{H-1} \frac{1}{\lambda_m^0 + h}.$$

Both sums can be arranged into alternating sums with absolutely decreasing terms. Consequently we get

$$\left| \sum_{h=-H}^{H-1} \sin^2 \frac{\pi}{2} (\lambda_m^0 + h) \frac{1}{\lambda_m^0 + h} \right| \leqslant \pi.$$

For $y_1 = 0$ and $\lambda_m^0 = 0$ we have

$$\left| \sum_{h=-H}^{H-1} {}^* \sin^2 \left(\frac{\pi}{2} h \right) \frac{1}{h} \right| = \left| \sum_{\substack{h \geqslant -H \\ h \equiv 1 (\mathrm{mod}\ 2)}}^{H-1} \frac{1}{h} \right| \leqslant \frac{1}{H}.$$

If $y_1 > 0$ then by Lemma 8 we may suppose that $\pi((\lambda_m^0 + H)/(2M+1))\, y_1 > \pi/2$, i.e., $H > (2M+1)/2y_1 - \lambda_m^0$. Let $H_0 = [(2M+1)/2y_1 - \lambda_m^0] + 1$. We have to show that

$$\left| \sum_{h=H_0}^{H-1} \sin^2 \frac{\pi}{2} (\lambda_m^0 + h) \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h} \right.$$

$$\left. + \sum_{h=-H+1}^{-H_0} \sin^2 \frac{\pi}{2} (\lambda_m^0 - 1 + h) \frac{\cos \pi \dfrac{\lambda_m^0 - 1 + h}{2M+1} y_1}{\lambda_m^0 - 1 + h} \right|$$

$$= \left| \overset{+}{\sum}_m + \overset{-}{\sum}_m \right| = O\left(\frac{1}{M}\right)$$

with a constant not depending on m. Splitting $\sum_m^+$ as in the proof of Lemma 6 we get

$$\overset{+}{\sum}_m = \sin^2 \frac{\pi}{2} \lambda_m^0 \sum_{\substack{h \geqslant H_0 \\ h \equiv 0 \bmod 2}}^{H-1} \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h}$$

$$+ \cos^2 \frac{\pi}{2} \lambda_m^0 \sum_{\substack{h \geqslant H_0 \\ h \equiv 1 \bmod 2}}^{H-1} \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1} y_1}{\lambda_m^0 + h}$$

$$= \sin^2 \frac{\pi}{2} \lambda_m^0 \overset{0+}{\sum}_m + \cos^2 \frac{\pi}{2} \lambda_m^0 \overset{1+}{\sum}_m,$$

$$\left| \overset{i+}{\sum}_m - \frac{1}{2} \int_{\tau_1}^{\tau_2} \frac{\cos t}{t} \, dt \right| \leqslant \frac{1}{2} w_{\pi/2} \left(\frac{2\pi y_1}{2M+1} \right) \pi \left(\frac{\lambda_m^0 + H - 1}{2M+1} y_1 - \frac{1}{2} \right) + \frac{2y_1}{2M+1},$$

where

$$\tau_1 = \frac{\pi}{2},$$

$$\tau_2 = \pi \frac{\lambda_m^0 + H - 1}{2M+1} y_1.$$

Applying a similar splitting to $\sum_m^-$ we obtain

$$\left| \overset{+}{\sum}_m + \overset{-}{\sum}_m \right| \leqslant \frac{1}{2} w_{\pi/2} \left(\frac{2\pi y_1}{2M+1} \right) \pi \frac{2H-1}{2M+1} y_1 + \frac{4y_1}{2M+1}$$

$$\leqslant \frac{1}{2} w_{\pi/2} \left(\frac{2\pi y_1}{2M+1} \right) 2\pi \left(1 + \frac{1}{c} \right) y_1 + \frac{4y_1}{2M+1}. \quad \blacksquare$$

For the next assertion which gives an estimate for the sum over the points in G_2 (17) recall the definition (4) of $\bar{M}$.

LEMMA 10.

$$\left| \frac{2}{\pi^2} \sum_{m=0}^{\bar{M}} \frac{\cos \pi \frac{2m+1}{2M+1} x_1}{2m+1} \sum_{\substack{l=-(2M+1) \\ l \neq -\lambda_m}}^{2M+1-2[\lambda_m]-1} \sin^2 \frac{\pi}{2} (\lambda_m + l) \frac{\cos \pi \frac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} \right| < \varepsilon$$

for all sufficiently large M.

Proof. Choosing $\eta \in]0, 1[\cap]0, (1/2y_1)[\cap]0, c[$, by Lemma 6 we have with a constant K not depending on m

$$\begin{aligned} &\left| \frac{2}{\pi^2} \sum_{m=0}^{\min(M, (1-\eta) M_c)} \frac{\cos \pi \frac{2m+1}{2M+1} x_1}{2m+1} \right. \\ &\quad \left. \times \sum_{l=-(2M+1)}^{2M+1-2[\lambda_m]-1} * \sin^2 \frac{\pi}{2} (\lambda_m + l) \frac{\cos \pi \frac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} \right| \\ &\leqslant \frac{2}{\pi^2} \sum_{0 \leqslant m \leqslant \min(M, (1-\eta) M_c)} \frac{1}{2m+1} \frac{K}{M} \\ &\leqslant (2 + \log(2M+1)) \frac{K}{M} \\ &= o(1) \qquad \text{as} \quad M \to \infty. \end{aligned}$$

If $c > 1$, then for sufficiently large M and suitably small η one has $M < (1-\eta)[M_c]$ and the assertion is already proved. If $c \leqslant 1$, then by Lemma 9 we get

$$\begin{aligned} &\left| \frac{2}{\pi^2} \sum_{(1-\eta) M_c < m \leqslant M_c} \frac{\cos \pi \frac{2m+1}{2M+1} x_1}{2m+1} \right. \\ &\quad \left. \times \sum_{l=-(2M+1)}^{2M+1-2[\lambda_m]-1} * \sin^2 \frac{\pi}{2} (\lambda_m + l) \frac{\cos \pi \frac{\lambda_m + l}{2M+1} y_1}{\lambda_m + l} \right| \\ &\leqslant \frac{2}{\pi^2} \eta M_c \frac{6 + O\left(\frac{1}{M}\right)}{2(1-\eta) M_c} \\ &\leqslant \frac{6\eta}{\pi^2(1-\eta)} + O\left(\frac{1}{M}\right). \end{aligned}$$

Choosing η sufficiently small we obtain the assertion. ∎

Proof of Assertion (e_1). By Lemmas 4, 5, and 10 it suffices to show that for $M \to \infty$ the sum over all points of the domain $F = G \setminus (G_1 \cup G_2 \cup G_3)$ as in (15), (16), (17), (18) comes arbitrarily close to the indicated limit if the parameter $\eta > 0$ determining the size of G_1 and G_3 is chosen sufficiently small. To this end we choose $\eta \in]0, 1[\cap]0, (1/2y_1)[$ and, in the case $c > 1$, we also choose $\eta < 1 - 1/c$ to begin with and we again decompose F into three subdomains F_1, F_2, F_3 as follows (if $c > 1$ and consequently $(1-\eta)\, c > 1$ the domains F_2 and F_3 are understood to be empty):

$$\begin{aligned}
&F_1 \cdots \eta < \mu \leqslant \min(1, (1-\eta)\, c),\\
&\qquad \alpha_1(\mu) = 1 - \frac{\mu}{c} \leqslant \nu \leqslant 1 + \frac{\mu}{c} = \beta_1(\mu);\\
&F_2 \cdots (1-\eta)\, c < \mu \leqslant \min(1, (1+\eta)\, c),\\
&\qquad \alpha_2 = \eta \leqslant \nu \leqslant 1 + \frac{\mu}{c} = \beta_2(\mu);\\
&F_3 \cdots (1+\eta)\, c < \mu \leqslant 1,\\
&\qquad \alpha_3(\mu) = \frac{\mu}{c} - 1 \leqslant \nu \leqslant \frac{\mu}{c} + 1 = \beta_3(\mu).
\end{aligned} \tag{21}$$

For each of these subdomains we can estimate the value of the corresponding sum

$$\begin{aligned}
\sigma_{m,i} &= \sum_{\alpha_i \leqslant h/(2M+1) \leqslant \beta_i} \sin^2 \frac{\pi}{2} (\lambda_m^0 + h)\, \frac{\cos \pi \dfrac{\lambda_m^0 + h}{2M+1}\, y_1}{\lambda_m^0 + h}\\
&= \sin^2 \frac{\pi}{2} \lambda_m^0 \sum_{\alpha_i \leqslant 2h/(2M+1) \leqslant \beta_i} \frac{\cos \pi \dfrac{\lambda_m^0 + 2h}{2M+1}\, y_1}{\lambda_m^0 + 2h}\\
&\quad + \cos^2 \frac{\pi}{2} \lambda_m^0 \sum_{\alpha_i \leqslant (2h-1)/(2M+1) \leqslant \beta_i} \frac{\cos \pi \dfrac{\lambda_m^0 + 2h - 1}{2M+1}\, y_1}{\lambda_m^0 + 2h - 1}\\
&= \sin^2 \left(\frac{\pi}{2} \lambda_m^0\right) \sigma'_{m,i} + \cos^2 \left(\frac{\pi}{2} \lambda_m^0\right) \sigma''_{m,i}.
\end{aligned}$$

We shall do this in some detail only for $i = 1$ and $c \leqslant 1$; the reasoning in the other cases is similar and left to the reader.

Since we want to respect the limits on m and l appearing in Lemmas 4, 5, and 10, the actual summation limits corresponding to the sets

$$
\begin{aligned}
F_1\colon & \eta M < m \leqslant \min(M, (1-\eta) M_c),\\
& \quad 2M+1-2[\lambda_m] \leqslant l \leqslant 2M+1;\\
F_2\colon & (1-\eta) M_c < m \leqslant \min(M, (1+\eta) M_c),\\
& \quad \eta(2M+1)-\lambda_m < l \leqslant 2M+1;\\
F_3\colon & (1+\eta) M_c < m \leqslant M,\\
& \quad -(2M+1) \leqslant l \leqslant 2M+1,
\end{aligned}
$$

will again differ slightly from those indicated in (21). Note, e.g., that the sets $\{m\colon \eta < (2m+1)/(2M+1) \leqslant (1-\eta)\, c\}$ and $\{m\colon \eta M < m \leqslant (1-\eta)\, M_c\}$ differ about at most two values of m which contribute $O(1/M)$ to the estimate of $s_M^{(6)}$. As a consequence, in order to find the limiting behaviour of $s_M^{(6)}$ either set of inequalities may be used in the definition of the set F_1. Analogous statements are true for the bounds α_i, β_i and for the sets F_2 and F_3.

An estimate of $\sigma'_{m,1}$ is furnished by the Riemann sum lemma:

$$
\left| \frac{1}{2} \sum_{2M+1-[\lambda_m] \leqslant 2h \leqslant 2M+1+[\lambda_m]} \frac{\cos \pi \dfrac{\lambda_m^0+2h}{2M+1}\, y_1}{\pi \dfrac{\lambda_m^0+2h}{2M+1}\, y_1} \frac{2\pi y_1}{2M+1} - \frac{1}{2} \int_{\tau_1}^{\tau_2} \frac{\cos s}{s}\, ds \right|
$$

$$
\leqslant w_\alpha \left(\frac{2\pi y_1}{2M+1} \right) 2\pi y_1 + O\left(\frac{1}{M}\right) = O\left(\frac{1}{M}\right),
$$

where

$$
\tau_1 = \pi y_1 \left(1 - \frac{1}{c} \frac{2m+1}{2M+1}\right),
$$

$$
\tau_2 = \pi y_1 \left(1 + \frac{1}{c} \frac{2m+1}{2M+1}\right),
$$

$$
\alpha = \frac{\pi y_1 \eta}{2},
$$

and where the constants in $O(1/M)$ depend on η but not on m. Writing

$$
g_1(t) = \int_{\pi y_1 - (t\pi y_1/c)}^{\pi y_1 + (t\pi y_1/c)} \frac{\cos s}{s}\, ds \qquad (0 \leqslant t \leqslant c(1-\eta))
$$

we therefore have

$$
\left| \sigma'_{m,1} - \frac{1}{2} g_1\left(\frac{2m+1}{2M+1}\right) \right| \leqslant w_\alpha \left(\frac{2\pi y_1}{2M+1}\right) 2\pi y_1 + O\left(\frac{1}{M}\right) = O\left(\frac{1}{M}\right).
$$

Essentially the same inequality holds for $\sigma''_{m,1}$ and therefore also for $\sigma_{m,1}$. The contribution of the summands of $s_M^{(6)}$ corresponding to points $P(m,h) \in F_1$ may now be estimated by

$$\left| \frac{2}{\pi^2} \sum_{m>\eta M}^{\min(M,(1-\eta)M_c)} \frac{\cos \pi \frac{2m+1}{2M+1} x_1}{2m+1} \right.$$

$$\times \sum_{l=2M+1-2[\lambda_m]}^{2M+1} \sin^2 \frac{\pi}{2}(\lambda_m+l) \frac{\cos \pi \frac{\lambda_m+l}{2M+1} y_1}{\lambda_m+l}$$

$$\left. -\frac{1}{\pi^2} \sum_{m>\eta M}^{\min(M,(1-\eta)M_c)} \frac{\cos \pi \frac{2m+1}{2M+1} x_1}{2m+1} g_1\left(\frac{2m+1}{2M+1}\right) \right|$$

$$\leqslant \frac{2}{\pi^2} \frac{M}{2\eta M} \cdot O\left(\frac{1}{M}\right) = O\left(\frac{1}{M}\right).$$

On the other hand, for $x_1 > 0$ we also have (by always the same Riemann sum reasoning)

$$\left| \frac{1}{2} \sum_{m>\eta M}^{\min(M,(1-\eta)M_c)} \frac{\cos \pi \frac{2m+1}{2M+1} x_1}{\pi \frac{2m+1}{2M+1} x_1} g_1\left(\frac{2m+1}{2M+1}\right) \frac{2\pi x_1}{2M+1} \right.$$

$$\left. -\frac{1}{2} \int_{\pi x_1 \eta}^{\pi x_1 \min(1,(1-\eta)c)} \frac{\cos t}{t}\, dt \int_{\pi y_1 - (t y_1 / c x_1)}^{\pi y_1 + (t y_1 / c x_1)} \frac{\cos s}{s}\, ds \right| = O\left(\frac{1}{M}\right).$$

Combining these estimates we get

$$\left| \frac{2}{\pi^2} \sum_{m>\eta M}^{\min(M,(1-\eta)M_c} \frac{\cos \pi \frac{2m+1}{2M+1} x_1}{2m+1} \right.$$

$$\times \sum_{l=2M+1-2[\lambda_m]}^{2M+1} \sin^2 \frac{\pi}{2}(\lambda_m+l) \frac{\cos \pi \frac{\lambda_m+l}{2M+1} y_1}{\lambda_m+l}$$

$$\left. -\frac{1}{2\pi^2} \int_{\pi x_1 \eta}^{\pi x_1 \min(1,(1-\eta)c)} \frac{\cos t}{t}\, dt \int_{\pi y_1 - (t y_1 / c x_1)}^{\pi y_1 + (t y_1 / c x_1)} \frac{\cos s}{s}\, ds \right| = O\left(\frac{1}{M}\right).$$

Thus we have checked assertion (e_1) as far as summation over terms corresponding to points in F_1 is concerned. Similar reasonings furnish estimates of the contribution to $s_M^{(6)}$ of terms corresponding to points in F_2

$$\left| \frac{2}{\pi^2} \sum_{m>(1-\eta)M_c}^{\min(M,(1+\eta)M_c)} \frac{\cos \pi \dfrac{2m+1}{2M+1} x_1}{2m+1} \right.$$

$$\times \sum_{l>\eta(2M+1)-\lambda_m}^{2M+1} \sin^2 \frac{\pi}{2}(\lambda_m+l) \frac{\cos \pi \dfrac{\lambda_m+l}{2M+1} y_1}{\lambda_m+l}$$

$$\left. -\frac{1}{2\pi^2} \int_{\pi x_1 \min(1,(1-\eta)c)}^{\pi x_1 \min(1,(1+\eta)c)} \frac{\cos t}{t} dt \int_{\eta\pi y_1}^{\pi y_1+(ty_1/cx_1)} \frac{\cos s}{s} ds \right|$$

$$= O\left(\frac{1}{M}\right)$$

(both terms of the difference between the absolute value signs, incidentally, may be made arbitrarily small for sufficiently small η) and to points of F_3 in case $(1+\eta) M_c < M$

$$\left| \frac{2}{\pi^2} \sum_{m>(1+\eta)M_c}^{M} \frac{\cos \pi \dfrac{2m+1}{2M+1} x_1}{2m+1} \sum_{l=-(2M+1)}^{2M+1} \sin^2 \frac{\pi}{2}(\lambda_m+l) \frac{\cos \pi \dfrac{\lambda_m+l}{2M+1} y_1}{\lambda_m+l} \right.$$

$$\left. -\frac{1}{2\pi^2} \int_{\pi x_1(1+\eta)c}^{\pi x_1} \frac{\cos t}{t} dt \int_{(ty_1/cx_1)-\pi y_1}^{(ty_1/cx_1)+\pi y_1} \frac{\cos s}{s} ds \right| = O\left(\frac{1}{M}\right).$$

The proof of assertion (e_1) is now completed by observing that for sufficiently large M the sum of the three first terms in the last three estimates by Lemmas 4, 5, and 10 differs arbitrarily little from $s_M^{(6)}$, while the sum of the three last terms in these estimates by Lemma 1 differs arbitrarily little from the limit indicated in the theorem. ∎

Proof of Assertion (e_2). The proof proceeds as the one of assertion (e_1), but the function g_1 has now to be replaced by

$$g_2(t) = \begin{cases} \displaystyle\int_{1-(t/c)}^{1+(t/c)} \frac{ds}{s} = \log\left(\frac{c+t}{c-t}\right) & \text{for} \quad t < (1-\eta)\,c, \\ \displaystyle\int_{\eta}^{1+(t/c)} \frac{ds}{s} = \log\left(\frac{c+t}{c\eta}\right) & \text{for} \quad (1-\eta)\,c \leqslant t \leqslant (1+\eta)\,c, \\ \displaystyle\int_{(t/c)-1}^{t/c)+1} \frac{ds}{s} = \log\left(\frac{t+c}{t-c}\right) & \text{for} \quad c < t. \quad \blacksquare \end{cases}$$

The proofs of assertions (e_3) and (e_4) proceed as the proofs of assertions (e_1) and (e_2) with suitable replacements of the involved integrands.

3. LOCALIZATION

The fact that the studied corner point Gibbs phenomenon is locally determined follows from the following result.

THEOREM 2. *Let the set A be as in Section 1. For $\rho \leqslant \pi$ let B denote the (open or closed) disk in $\mathbb{R}^2$ with radius ρ about the point $(0, 0)$ and let g denote the indicator function of the set $C = A \backslash B$, extended periodically with period 2π in x and y. Then for the partial sum of the Fourier series of g*

$$S_{n,n}\left(\frac{x}{n}, \frac{y}{n}; g\right) = \sum_{k=-n}^{n} \sum_{l=-n}^{n} \hat{g}_{k,l} e^{2\pi i (k(x/n) + l(y/n))}$$

one has

$$\lim_{n \to \infty} S_{n,n}\left(\frac{x}{n}, \frac{y}{n}; g\right) = 0$$

uniformly for (x, y) in any bounded domain D of $\mathbb{R}^2$.

Proof. Let

$$D_n(s) = \frac{\sin\left(n + \dfrac{1}{2}\right)s}{2 \sin \dfrac{s}{2}}.$$

For simplicity we assume $1 \leqslant c < \infty$; the remaining cases may be treated similarly. The boundary of B intersects the line $t = cs$ in the point $(s_0, t_0) = (\rho/\sqrt{1+c^2}, c\rho/\sqrt{1+c^2})$. We decompose C into the sets

$$C_1 = \{(s, t) \in C : t \leqslant t_0\},$$

$$C_2 = C \backslash C_1$$

and we denote by g_1 and g_2 the periodically extended indicator functions of C_1 and C_2, respectively.

For $(x, y) \in D$ we have

$$S_{n,n}\left(\frac{x}{n}, \frac{y}{n}; g_2\right) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g_2\left(\frac{x}{n} + s, \frac{y}{n} + t\right) D_n(s)\, D_n(t)\, ds\, dt$$

$$= \frac{1}{\pi^2} \iint_{(s,t) \in C_2 - (x/n,\, y/n)} D_n(s)\, D_n(t)\, ds\, dt.$$

We suppose n to be so large, that $|x/n| < s_0/2$ and $|y/n| < t_0/2$ for all $(x, y) \in D$. In the double integral s varies over the interval $[s_0 - x/n, \pi(1 + 1/c) - x/n] \subset [s_0/2, 2\pi + s_0/2]$, while for fixed s the variable t varies over an interval $[t_1(s, x, y, n), t_2(s, x, y, n)] \subset T = [t_0/2, \pi + t_0/2]$. Since the function $1_T(t)/\sin(t/2)$ has finite variation we have for a constant G only depending on t_0

$$\left| \int_{t_1(s,x,y,n)}^{t_2(s,x,y,n)} D_n(t)\, dt \right| \leqslant \frac{G}{n}$$

and by [16, Chap. II, 12.1]

$$\left| S_{n,n}\left(\frac{x}{n}, \frac{y}{n}; g_2\right) \right| \leqslant \frac{G}{n\pi^2} \int_{s_0/2}^{2\pi + s_0/2} |D_n(s)|\, ds$$

$$\leqslant \frac{G}{n\pi}\left(\frac{4}{\pi^2} \log n + O(1)\right).$$

A similar reasoning with interchanged roles of s and t applies to $S_{n,n}(x/n, y/n; g_1)$. ∎

Obviously the reasoning above may also applied if the disk B is replaced by some suitable other non-circular (e.g., any convex) neighbourhood of the point $(0, 0)$.

4. BEHAVIOUR OF THE GIBBS PHENOMENON FOR $c \searrow 0$ AND FOR $c \nearrow 0$

Our last goal is the study of the behaviour of the function $s(x_1, y_1, c)$ of Eq. (5) as $c \searrow 0$ and $c \nearrow 0$. Observe that for $c \to 0$ the variable x_1 as in (1) looses its quality as a coordinate. We shall therefore write

$$x = \frac{\pi x_0}{2M+1}, \qquad y = \frac{\pi y_0}{2M+1}$$

such that

$$x_1 = x_0 - \frac{y_0}{c}, \qquad y_1 = y_0.$$

For the integral in assertion (b) of Theorem 1 we obtain

$$\lim_{c \searrow 0} \frac{1}{2\pi} \int_0^{\pi\bar{c}(x_0 - y_0/c)} \frac{\sin t}{t}\, dt = -\frac{1}{2\pi} \int_0^{\pi y_0} \frac{\sin t}{t}\, dt, \tag{22}$$

$$\lim_{c \nearrow 0} \frac{1}{2\pi} \int_0^{\pi\bar{c}(x_0 - y_0/c)} \frac{\sin t}{t}\, dt = \frac{1}{2\pi} \int_0^{\pi y_0} \frac{\sin t}{t}\, dt. \tag{23}$$

In order to evaluate the integrals $I^{(5)}$ and $I^{(6)}$ in assertions (d) and (e) for $c \searrow 0$ and for $c \nearrow 0$ we shall first rewrite them applying the substitutions

$$t = \begin{cases} \pi c x_1 u & \text{for} \quad x_1 \neq 0, \\ c u & \text{for} \quad x_1 = 0, \end{cases}$$

$$s = \pi y_1 v \qquad \text{for} \quad y_1 \neq 0.$$

Note that for the study of the limiting behaviour of the integrals as $c \to 0$ we may suppose $x_1 = x_0 - y_0/c \neq 0$ except for $x_1 = x_0 = y_1 = y_0 = 0$. For $c \neq 0$ and $x_1 \neq 0$, $y_1 = y_0 \neq 0$ we have

$$\begin{aligned} I^{(5)}(x_0, y_0, c) &= \frac{1}{2\pi^2} \int_0^{\pi x_1} \frac{\sin t}{t}\, dt \int_{(ty_1/cx_1) - \pi y_1}^{(ty_1/cx_1) + \pi y_1} \frac{\sin s}{s}\, ds \\ &= \frac{1}{2\pi^2} \int_0^{1/c} \frac{\sin \pi(cx_0 - y_0)\, u}{u}\, du \int_{u-1}^{u+1} \frac{\sin \pi y_0 v}{v}\, dv, \end{aligned} \tag{24}$$

and for $1 > c > 0$ and $x_1 \neq 0$, $y_1 \neq 0$

$$\begin{aligned} I^{(6)}(x_0, y_0, c) = &-\frac{1}{2\pi^2} \left\{ \int_0^{\pi x_1 c} \frac{\cos t}{t}\, dt \int_{\pi y_1 - (ty_1/cx_1)}^{\pi y_1 + (ty_1/cx_1)} \frac{\cos s}{s}\, ds \right. \\ &\left. + \int_{\pi x_1 c}^{\pi x_1} \frac{\cos t}{t}\, dt \int_{(ty_1/cx_1) - \pi y_1}^{(ty_1/cx_1) + \pi y_1} \frac{\cos s}{s}\, ds \right\} \\ = &-\frac{1}{2\pi^2} \left\{ \int_0^1 \frac{\cos \pi(cx_0 - y_0)\, u}{u}\, du \int_{1-u}^{1+u} \frac{\cos \pi y_0 v}{v}\, dv \right. \\ &\left. + \int_0^{1/|c|} \frac{\cos \pi(cx_0 - y_0)\, u}{u}\, du \int_{u-1}^{u+1} \frac{\cos \pi y_0 v}{v}\, dv \right\}. \end{aligned} \tag{25}$$

The last expression is also valid for $y_0 = 0$ resp. for $x_0 = y_0 = 0$ while it simply changes its sign for $-1 < c < 0$. At this point we may already observe that for $x_1 = x_0 = y_1 = y_0 = 0$ the integrals in Theorem 1 on the right sides of assertions (a), (b), (c), and (d) vanish while for $c \searrow 0$ the absolute value of $s^{(6)}(0, 0, c)$ increases, i.e., $s(0, 0, c)$ is decreasing for $c \searrow 0$ and increasing for $c \nearrow 0$.

As functions of u all integrands above are dominated by the function

$$h(u) = \frac{1}{2u} \log \left(\frac{1+u}{1-u}\right)^2.$$

The integrability of h over $[0, 1]$ is already guaranteed by Lemma 1. It is also implied by

$$\begin{aligned} 0 \leqslant h(u) &= \frac{\log(1+u) - \log(1-u)}{u} \\ &= \frac{1}{u} \log \left(1 + \frac{2u}{1-u}\right) < \frac{1}{u} \cdot \frac{2u}{1-u} = \frac{2}{1-u} \qquad \text{for} \quad 0 < u < 1 \end{aligned}$$

(the first line implies integrability to the left of $u = 1$, the second line implies integrability to the right of $u = 0$). For $1 < u < \infty$ and $u = 1/w$ we have $1 > w > 0$ and

$$\frac{du}{2u} \log \left(\frac{1+u}{1-u}\right)^2 = -\frac{dw}{2w} \log \left(\frac{1+w}{1-w}\right)^2.$$

Consequently we have

$$\begin{aligned} \lim_{c \searrow 0} I^{(6)}(0, 0, c) &= -\frac{1}{2\pi^2} \int_0^\infty \frac{du}{2u} \log \left(\frac{1+u}{1-u}\right)^2 \\ &= -\frac{1}{\pi^2} \int_0^1 \frac{du}{u} \log \left(\frac{1+u}{1-u}\right) \\ &= -\frac{2}{\pi^2} \int_0^1 \sum_{k=0}^\infty \frac{u^{2k}}{2k+1} \, du \\ &= -\frac{1}{4}, \end{aligned} \tag{26}$$

$$\lim_{c \searrow 0} s(0, 0, c) = 0,$$

$$\lim_{c \nearrow 0} s(0, 0, c) = \frac{1}{2}.$$

By Lebesgue's theorem on dominated convergence we may take the limit for $c \to 0$ under the integral signs in (24) and (25). This furnishes the following formulas in which the right sides are independent of x_0:

$$\lim_{c \searrow 0} I^{(5)}(x_0, y_0, c) = -\frac{1}{2\pi^2} \int_0^\infty \frac{\sin \pi y_0 u}{u}\, du \int_{u-1}^{u+1} \frac{\sin \pi y_0 v}{v}\, dv, \tag{27}$$

$$\lim_{c \searrow 0} I^{(6)}(x_0, y_0, c) = -\frac{1}{2\pi^2} \int_0^\infty \frac{\cos \pi y_0 u}{u}\, du \int_{|1-u|}^{1+u} \frac{\cos \pi y_0 v}{v}\, dv. \tag{28}$$

Again for $c \nearrow 0$ the sign of the double integral has to be changed. In order to evaluate the integral (28) we apply the substitution

$$w = u + v,$$

$$z = u - v.$$

We obtain

$$-\frac{1}{2\pi^2} \int_0^\infty \frac{\cos \pi y_0 u}{u}\, du \int_{|1-u|}^{1+u} \frac{\cos \pi y_0 v}{v}\, dv$$

$$= -\frac{1}{\pi^2} \int_{w=1}^\infty \int_{z=-1}^1 \frac{\cos \pi y_0 \frac{w+z}{2} \cos \pi y_0 \frac{w-z}{2}}{(w+z)(w-z)}\, dz\, dw \tag{29}$$

$$= -\frac{1}{4\pi^2} \left\{ \int_{w=1}^\infty 2\, \frac{\cos \pi y_0 w}{w} \log\left(\frac{w+1}{w-1}\right) dw \right.$$

$$\left. + \int_{z=-1}^1 \frac{\cos \pi y_0 z}{z} \log\left(\frac{1+z}{1-z}\right) dz \right\}$$

$$= -\frac{1}{4\pi^2} \int_0^\infty \frac{\cos \pi y_0 w}{w}, \log\left(\frac{1+w}{1-w}\right)^2 dw$$

$$= -\frac{1}{4} + \frac{1}{2\pi} \int_0^{\pi |y_0|} \frac{\sin t}{t}\, dt. \tag{30}$$

(The last equality is due to an application of formula 4.425 in [7] with $a = \pi\, |y_0|$ and $b = 1$.) Consequently we also obtain

$$\lim_{c \nearrow 0} I^{(6)}(x_0, y_0, c) = \frac{1}{4} - \frac{1}{2\pi} \int_0^{\pi\, |y_0|} \frac{\sin t}{t}\, dt. \tag{31}$$

In order to evaluate the integral (27) we use assertion (a) of the following lemma, which has been communicated to me by John Boersma (TU

Eindhoven). Assertion (b) of this lemma could also have been used in place of the reference to [7] in order to prove (30).

LEMMA 11. *For $a>0$ and $b>0$ one has*

$$\text{(a)}\quad \int_0^\infty \frac{\sin au}{u}\,du \int_{u-1}^{u+1} \frac{\sin bv}{v}\,dv = \pi \int_0^{\min(a,b)} \frac{\sin t}{t}\,dt,$$

$$\text{(b)}\quad \int_0^\infty \frac{\cos au}{u}\,du \int_{|u-1|}^{u+1} \frac{\cos bv}{v}\,dv = \frac{\pi^2}{2} - \pi \int_0^{\max(a,b)} \frac{\sin t}{t}\,dt.$$

Proof. Denote the double integral in the left member of assertion (a) by $I(a, b)$. Then for $t>0$ we have

$$\begin{aligned}
\frac{\partial I}{\partial t}(a, t) &= \int_0^\infty \frac{\sin au}{u}\,du \int_{u-1}^{u+1} \cos tv\,dv \\
&= \frac{1}{t}\int_0^\infty \frac{\sin au}{u}\,(\sin t(u+1) - \sin t(u-1))\,du \\
&= 2\frac{\sin t}{t}\int_0^\infty \frac{\sin au \cos tu}{u}\,du \\
&= \frac{\sin t}{t}\int_0^\infty \frac{\sin(a+t)\,u + \sin(a-t)\,u}{u}\,du \\
&= \begin{cases} \pi\dfrac{\sin t}{t} & \text{for} \quad 0<t<a, \\[2mm] \dfrac{\pi}{2}\dfrac{\sin t}{t} & \text{for} \quad t=a, \\[2mm] 0 & \text{for} \quad 0<a<t, \end{cases}
\end{aligned}$$

$$\begin{aligned}
I(a, b) &= I(a, 0) + \int_0^b \frac{\partial I}{\partial t}(a, t)\,dt \\
&= \begin{cases} \pi\displaystyle\int_0^b \frac{\sin t}{t}\,dt & \text{for} \quad 0<b<a, \\[2mm] \pi\displaystyle\int_0^a \frac{\sin t}{t}\,dt & \text{for} \quad 0<a\leqslant b. \end{cases}
\end{aligned}$$

This proves assertion (a). For the proof of assertion (b) denote the double integral in the corresponding left member by $J(a, b)$. Similarly as above for $a\geqslant 0$, $b\geqslant 0$ and $t>0$ we have

$$\begin{aligned}\frac{\partial J}{\partial t}(a, t) &= \int_0^\infty \frac{\cos au}{u}\, du \int_{|u-1|}^{u+1} (-\sin tv)\, dv \\ &= -2\,\frac{\sin t}{t} \int_0^\infty \frac{\cos au \sin tu}{u}\, du \\ &= \begin{cases} -\pi \dfrac{\sin t}{t} & \text{for} \quad 0 \leqslant a < t, \\ -\dfrac{\pi}{2} \dfrac{\sin t}{t} & \text{for} \quad 0 < t = a, \\ 0 & \text{for} \quad 0 < t < a. \end{cases}\end{aligned}$$

Note that the reasoning employed in (29) shows that $J(a, b) = J(b, a)$ and that by the computation of the integral in (26) we have

$$J(0, 0) = \int_0^\infty \frac{du}{2u} \log \left(\frac{1+u}{1-u}\right)^2 = \frac{\pi^2}{2}.$$

Therefore we get

$$\begin{aligned}J(a, b) &= J(0, 0) + \int_0^a \frac{\partial J}{\partial t}(t, 0)\, dt + \int_0^b \frac{\partial J}{\partial t}(a, t)\, dt \\ &= \frac{\pi^2}{2} - \pi \int_0^a \frac{\sin t}{t}\, dt - \begin{cases} 0 & \text{for} \quad b \leqslant a, \\ \pi \displaystyle\int_a^b \frac{\sin t}{t}\, dt & \text{for} \quad a < b. \end{cases} \quad \blacksquare\end{aligned}$$

Observe now that the integral in (27) is even as a function of y_0. Putting $a = b = \pi\, |y_0|$ we obtain

$$\lim_{c \searrow 0} I^{(5)}(x_0, y_0, c) = -\frac{1}{2\pi} \int_0^{\pi |y_0|} \frac{\sin t}{t}\, dt, \tag{32}$$

$$\lim_{c \nearrow 0} I^{(5)}(x_0, y_0, c) = \frac{1}{2\pi} \int_0^{\pi |y_0|} \frac{\sin t}{t}\, dt. \tag{33}$$

Adding up the right sides of (22), (30) and (32) resp. (23), (31), and (33) with $\frac{1}{4}$ and the right member of assertion (a) in Theorem 1 we obtain the following assertion.

THEOREM 3. *For $c \neq 0$ let*

$$S(x_0, y_0, c) = s\left(x_0 - \frac{y_0}{c}, y_0, c\right) = \lim_{M \to \infty} s_M\left(x_0 - \frac{y_0}{c}, y_0, c\right).$$

Then

$$\lim_{c \searrow 0} S(x_0, y_0, c) = 0,$$

$$\lim_{c \nearrow 0} S(x_0, y_0, c) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi y_0} \frac{\sin t}{t}\, dt.$$

Heuristically this corresponds with the fact that in the shrinking neighbourhood of 0 for $c \searrow 0$ the climb of the partial sums of the Fourier series of 1_A from approximately 0 to approximately 1 is shifted more and more to the right away from the origin, while for $c \nearrow 0$ it more and more resembles the climb from approximately 0 in the negative y-halfplane to approximately 1 in the positive y-halfplane described by the one-dimensional Gibbs phenomenon.

5. ADDITIONAL REMARKS

Remark 1. The integration domain in case (d) of Theorem 1 is a parallelogram. The integration domain in case (e) reduces for $c \geqslant 1$ to a triangle. For $c \to \infty$ the integrals in (e) vanish while the integral in (b) turns into

$$\frac{1}{2\pi} \int_0^{\pi x_1} \frac{\sin t}{t}\, dt$$

and the integral in (d) turns into

$$\frac{1}{\pi^2} \int_0^{\pi x_1} \frac{\sin t}{t}\, dt \int_0^{\pi y_1} \frac{\sin s}{s}\, ds.$$

This agrees with the limiting equation

$$s(x, y, \infty) = \left(\frac{1}{2} + \frac{1}{\pi} \int_0^{\pi x} \frac{\sin t}{t}\, dt \right) \left(\frac{1}{2} + \frac{1}{\pi} \int_0^{\pi y} \frac{\sin s}{s}\, ds \right)$$

obtained directly by exploiting the Gibbs phenomenon for the product of the indicator functions $1_{[0,\pi]}(x)$ and $1_{[0,\pi]}(y)$ extended with period 2π in both variables.

Remark 2. If the Gibbs phenomenon is to be established at the discontinuity in $(0, 0)$ using the indicator function of a set having there a corner point of the type

$$\{(x, y): c_1 x < y < c_2 x, 0 < x\},$$

then the limit function of the M-th partial sum in $(\pi x_0/(2M+1), \pi y_0/(2M+1))$ will be the difference of the limit functions corresponding to the sets (in case of positive constants c_i)

$$A_i = \left\{(x, y): 0 \leqslant y \leqslant \pi, \frac{y}{c_i} \leqslant x \leqslant \frac{y}{c_i} + \pi\right\} \qquad (i = 1, 2).$$

In each of these limit functions one has to put $x_1 = x_0 - y_0/c_i$, $y_1 = y_0$ respectively $(i = 1, 2)$.

Remark 3. A two-dimensional Gibbs' phenomenon at a corner point has already been studied by Weyl [14, 15]. He considers the indicator function of the complement of a region A on the sphere bounded by two meridians including an angle α. This function is developed into a series of spherical functions with I_n as the partial sum of order n. Denote by θ resp. ϕ the distance of a point P from the north pole resp. its geographical length. Furthermore, let ϕ_1 and ϕ_2 resp. be the difference between the geographical lengths of P and of the two meridians. As Weyl shows, one has $I_n(\theta, \phi) = \text{Ang}^{(\alpha)}(n\theta, \phi) + o(1)$ where $o(1) \to 0$ as $n \to \infty$ and $\theta \to 0$, uniformly in ϕ, and where

$$\text{Ang}^{(\alpha)}(n\theta, \phi) = \frac{1}{\pi^2} \int_{t=\theta}^{\infty} \frac{\sin nt}{t} \left\{ \arctan\left(\tan \phi_1 \sqrt{1 - \frac{\theta^2}{t^2}}\right) + \arctan\left(\tan \phi_2 \sqrt{1 - \frac{\theta^2}{t^2}}\right)\right\} dt.$$

Gibbs phenomena for functions on more-dimensional domains have also been studied by other authors [1–6, 8–11, 13] insofar as the inequality

$$\limsup_{\substack{n \to \infty \\ x \to x_0}} s_n(x) > \limsup_{x \to x_0} s(x)$$

or a similar inequality for lim inf has been established for various functions and types of discontinuities at x_0. The behaviour of the sequence of partial sums is investigated in direction of a normal to a curve along which a discontinuity occurs (there is also a considerable number of papers on summability of series expansions of such functions by methods under which the Gibbs phenomenon dissappears). In these papers, however, no attention is given to the possible existence of a limit of the approaching functions in correspondingly re-scaled neighbourhoods of a corner point x_0. In connection with [1, 13] it should also be noted that the discontinuous functions $f = 1_A$ of the present paper are for $0 < |c| < \infty$ not of the form $h(x) \cdot h(y)$ and not of bounded variation in the sense of Hardy and Krause.

Remark 4. The excess of the partial sum $s_M(x_1, y_1, c)$ above the level 1 depends on c. Taking into account the form of the integration domains for the limiting integrals in assertions (d) and (e) as well as the signs of $\cos t/t \cdot \cos s/s$ and $\sin t/t \cdot \sin s/s$ in corresponding subdomains one may roughly expect maxima for the limit function $s(x_1, y_1, c)$ (5) for $x_1 \approx 1$ and $y_1 \approx 1$ and of increasing size for increasing c. In the limit, for $c \to \infty$, by Remark 1 the maximum of s_M for $M \to \infty$ should approach $(1 + 0.1789797 \cdots /2)^2 \approx 1.187$ with an overshoot of $2(\max(s_M) - 1)\% \approx$ 37.4% of half the jump size. In fact, for $M = 20$ the following rounded values may be observed which experimentally appear as relative maxima:

c	x_0	$x_1 = x_0 - \frac{y_0}{c}$	$y_1 = y_0$	s_M	$2(s_M - 1)\%$
0.1	19.77	9.97	0.98	1.167	33.3%
0.5	3.96	2.00	0.98	1.183	36.7%
1	2.01	1.02	0.99	1.181	36.2%
2	1.45	0.96	0.99	1.178	35.7%
3	1.29	0.96	0.99	1.183	36.5%
4	1.21	0.96	0.99	1.185	36.9%
5	1.16	0.96	0.99	1.185	37.1%
10	1.07	0.97	0.99	1.187	37.3%
20	1.02	0.97	0.99	1.187	37.4%
35	1.00	0.97	0.98	1.187	37.4%
50	0.99	0.97	0.98	1.187	37.4%
100	0.98	0.97	0.98	1.187	37.4%

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